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Numerical Solution of Linear Difference Equations

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Daniel W. Lozier

Mathematical Analysis Division
Center for Applied Mathematics
National Engineering Laboratory
National Bureau of Standards
U.S. Department of Commerce
Washington, D.C. 20234

March 1980

Final Report



U.S. DEPARTMENT OF COMMERCE

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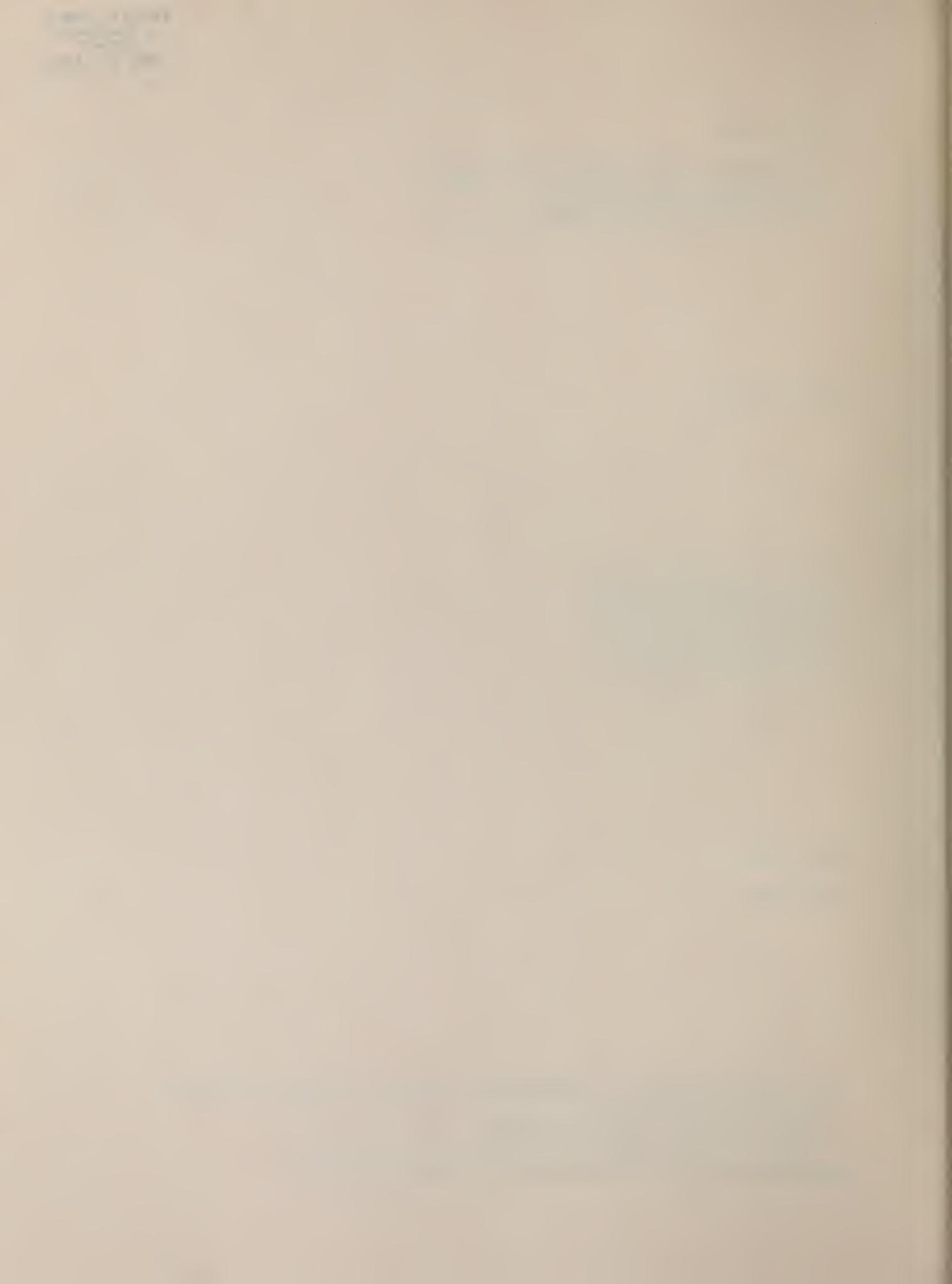
Final Report

U.S. DEPARTMENT OF COMMERCE, Philip M. Klutznick, Secretary

Luther H. Hodges, Jr., Deputy Secretary

Jordan J. Baruch, Assistant Secretary for Science and Technology

NATIONAL BUREAU OF STANDARDS, Ernest Ambler, Director



ABSTRACT

Consider a given homogeneous or inhomogeneous linear difference equation $\sum_{s=0}^{\ell} d_s(r)y(r+s) = g(r)$ where $\ell \geq 2$ and $r = 0, 1, 2, \dots$.

Suppose y is a solution of this equation and u, v are solutions of the homogeneous form of this equation such that $u(r)/v(r) \rightarrow 0$, $y(r)/v(r) \rightarrow 0$, $u(r)/y(r) \rightarrow 0$. It is known that under these circumstances algorithms for the computation of y based on forward recurrence or backward recurrence, such as the Miller algorithm, are numerically unstable.

Stable algorithms, such as the method of Olver in the case $\ell = 2$, have been based on approximating $y(r)$ by the solutions $y_n(r)$ of a certain sequence of boundary value problems. More specifically, $y_n(r)$ is a solution of the difference equation that coincides with $y(r)$, over some fixed initial range of r , say $r = i, i+1, \dots, i+j-1$, and satisfies $y_n(r) = 0$ for $r = n, n+1, \dots, n+\ell-j-1$. Here j is an integer whose value depends on the asymptotic behavior of the chosen solution $y(r)$ and n is an arbitrary large integer. Boundary value problems of this type are shown to be equivalent to two initial value problems of order j and $\ell-j$ by factorization of the linear difference operator. The solution of the problem of order j is obtained by forward recurrence; the solution of the other problem is obtained by backward recurrence.

The algorithm is specified completely for a broad class of linear difference operators. This class includes, for example, every constant-coefficient operator. Convergence of $y_n(r)$ to $y(r)$ as $n \rightarrow \infty$ for fixed r is proved and an expansion of the truncation error is derived. Numerical stability is demonstrated under appropriate conditions. The method is tested by numerical examples involving fourth-order equations with variable coefficients.

PREFACE

This paper was submitted as a Ph.D. thesis in the Interdisciplinary Applied Mathematics Program of the University of Maryland at College Park. The Ph.D. degree was awarded on December 18, 1979.

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CHAPTER 1. INTRODUCTION

1.1 Miller's and Olver's Algorithms

Linear recurrence relations (linear difference equations) are satisfied by many of the higher transcendental functions of mathematics, and they arise quite naturally in applications of mathematics. Discretization of ordinary differential equations in numerical analysis is one important area in which they occur. Because of their special form, linear recurrence relations are especially well suited to the construction of computing algorithms. Stability requirements in the case of second-order homogeneous relations led to the well-known Miller backward recurrence algorithm [5], which was developed for the computation of modified Bessel functions. This reference was published in 1952. The review paper of Gautschi [9] provides a survey of algorithms and applications in the case of second-order homogeneous equations. Nowadays, the name of Miller often attends when a minimal solution (defined below) of a homogeneous equation of arbitrary order is computed using backward recurrence [28]. These algorithms always involve backward recurrence from assumed starting values (since these are not known in general) followed by a normalization procedure, such as matching one value of the computed solution with the desired solution in order to arrive at the proper scale factor which relates these two solutions.

A different class of algorithms becomes necessary when "intermediate" solutions of a linear recurrence relation are to be computed. Intuitively speaking, an intermediate solution is one for which two other solutions can be found such that one dominates, and the other is dominated by, the intermediate solution. We are speaking here of dominance in the following sense:

if $a = (a(0), a(1), \dots)$ and $b = (b(0), b(1), \dots)$ are two infinite sequences⁺ then b dominates a whenever $a(r)/b(r) \rightarrow 0$ as $r \rightarrow \infty$.

Neither forward recurrence nor the Miller algorithm is numerically stable for an intermediate solution. This is because recurrence in either direction, when carried out in finite precision, will be contaminated by small components of the other solutions. An analysis of the propagation of these rounding errors appears in [18]. In the forward direction, as the recurrence proceeds through more and more steps, the dominant solutions grow and eventually overwhelm the desired intermediate solution. Since our concept of dominance is really dominance "at infinity", the situation for recurrence in the backward direction is not completely analogous. However, if we start at a sufficiently large value of r and recur backward, other solutions grow at a faster rate than the wanted intermediate solution as r decreases.

In addition to intermediate solutions, a difference equation has minimal (or "recessive") and maximal (or "dominant") solutions. Intuitively, a minimal solution does not dominate, and a maximal solution is not dominated by, any other solution of the difference equation as $r \rightarrow \infty$. Ultimately, forward recurrence is stable for maximal solutions since the error cannot grow more rapidly than the solution. Similarly, backward recurrence is stable for minimal solutions, at least in a region of sufficiently large r .

⁺We shall regard the solutions of a linear difference equation, as well as the coefficients of the equation, as complex-valued functions of a discrete real variable. We shall assume that this discrete real variable proceeds in unit increments; thus the solutions, as well as the coefficients, of linear difference equations are complex sequences. We shall be interested only in those linear difference equations whose coefficient sequences are of infinite length.

The lowest order of difference equation for which an intermediate solution can exist, and then only when the equation is inhomogeneous, is two.

Such an equation is of the form

$$(1.1.1) \quad a(r)y(r-1) - b(r)y(r) + c(r)y(r+1) = d(r), \quad r = 1, 2, \dots .$$

Since the general solution is

$$y(r) = \alpha u(r) + \beta v(r) + h(r), \quad r = 0, 1, 2, \dots ,$$

where α and β are constants, $u = (u(0), u(1), \dots)$ and $v = (v(0), v(1), \dots)$ are linearly independent solutions of the homogeneous equation, and $h = (h(0), h(1), \dots)$ is a particular solution of (1.1.1), there can be at most three asymptotically distinct types of behavior of solutions as $r \rightarrow \infty$. Let us assume that v dominates u and that h is an intermediate solution. In addition, assume $u(0) \neq 0$ and $v(r) \neq 0$ for all sufficiently large r . Under these assumptions a stable algorithm for computing intermediate solutions of (1.1.1) was presented by Olver in [21]. If $y = (y(0), y(1), \dots)$ is the desired intermediate solution, Olver's algorithm approximates y by a sequence of solutions $y_n = (y_n(0), y_n(1), \dots)$ such that $y_n(0) = y(0)$ and $y_n(n) = 0$ where n is an appropriately chosen positive integer. Note that only one initial value of the wanted intermediate solution is required to determine the sequence of approximations. The sequence is shown to converge pointwise to y as $n \rightarrow \infty$.

Olver's algorithm efficiently produces the first $n-1$ unknown terms of y_n as the solution of a tridiagonal linear system of order $n-1$. The procedure used for solving these linear equations is equivalent to Gaussian elimination without pivoting. It consists, in effect, of a forward elimination stage followed by a back substitution stage. The algorithm is constructed so that the numerical stability of this procedure may be assessed. The

forward elimination stage is realized by two forward recurrences. One of these uses the homogeneous form of (1.1.1) to generate a dominant solution $p = (p(0), p(1), \dots)$. The initial values used for starting this recurrence are $p(0) = 0$ and $p(1) = 1$, and the solution p is asymptotically proportional to v . Therefore, the rounding errors ultimately propagate in proportion to p , although they could propagate faster than p at first (if p is essentially proportional to u for early values of r , corresponding to an excessively small value of $u(0)$). A combined algorithm that employs both Miller's algorithm and Olver's algorithm in adjacent ranges of r is proposed in [22].

The second forward recurrence in the forward elimination stage of Olver's algorithm uses the inhomogeneous first-order equation

$$a(r)e(r-1) - c(r)e(r) = d(r)p(r), \quad r = 1, 2, \dots,$$

where the solution $e = (e(0), e(1), \dots)$ satisfies $e(0) = y(0)$. Here two distinct asymptotic forms of behavior may exist, but there is no reason to expect that the calculation of e will be asymptotically unstable in general. However, as was the case for the solution p , there may be applications where rounding errors grow more rapidly than $e(r)$ over some finite initial range of values of r .

The back substitution consists of backward recurrence of the inhomogeneous first-order equation

$$p(r+1)y_n(r) - p(r)y_n(r+1) = e(r), \quad r = n-1, n-2, \dots, 1,$$

starting with $y_n(n) = 0$. The stability of this backward recurrence is guaranteed in the "asymptotic region" belonging to the chosen solution $y(r)$, that is, the infinite range of r which is such that the asymptotic

behavior of $y(r)$ relative to $u(r)$ and $v(r)$ is maintained.

Olver's method of solving the tridiagonal system leads to a very convenient representation of the truncation error. For each r and for all $n \geq r$, the expansion

$$(1.1.2) \quad y(r) = y_n(r) + p(r) \sum_{s=n}^{\infty} \frac{e(s)}{p(s)p(s+1)}$$

is valid. Consequently, for each r and $n \geq r$ and $v \geq 1$ we have

$$y_{n+v}(r) - y_n(r) = p(r) \sum_{s=n}^{n+v-1} \frac{e(s)}{p(s)p(s+1)} .$$

This expansion is useful as a basis for deciding on an appropriate value of n , i.e., for deciding when to stop the forward elimination and begin the back substitution. Suppose that for particular values of $\varepsilon > 0$ and r , say $r = m$, it is desired that

$$|y(m) - y_n(m)| < \varepsilon .$$

A value of $v (\geq 1)$ is selected and $y(m)$ is approximated by $y_{n+v}(m)$.

The problem then is to determine the smallest value of n such that

$$|y_{n+v}(m) - y_n(m)| < \varepsilon .$$

This is done by continuing the forward elimination until $r = m$ at which point calculation of the numbers

$$\eta(r) = |p(m) \sum_{s=r}^{r+v-1} \frac{e(s)}{p(s)p(s+1)}| , \quad r = m, m+1, \dots$$

is begun. Forward elimination is continued further, together with the calculation of $\eta(r)$, until a value of r is reached such that $\eta(r) < \varepsilon$. This value of r is taken for n ; its adequacy may be checked by taking additional terms in the infinite expansion of $y(m) - y_n(m)$, given by (1.1.2) with $r = m$.

In addition to being appropriate for computing intermediate solutions of (1.1.1), Olver's algorithm is suited to computing the minimal solution of a homogeneous equation. It can therefore be compared to Miller's algorithm. The advantage of Olver's algorithm over Miller's algorithm is that the latter does not have a criterion for determining automatically an appropriate value for n . On the other hand, if an adequate value of n is known from other considerations (as, for example, from an asymptotic estimate), then Miller's algorithm may require less computational effort.

Zahar [27], Oliver [19,20], and Cash [6] have considered extensions of Olver's algorithm to linear difference equations of higher order. In the present work we shall construct and analyze an algorithm which has features in common with these extensions. We shall also illustrate the algorithm by means of numerical examples.

1.2 Factorization of Linear Difference Equations

Let \mathbb{F} be the field of complex numbers and let S be the set of all complex-valued infinite sequences $x = (x(0), x(1), \dots)$. Clearly S is an infinite-dimensional linear space over \mathbb{F} . Let \mathcal{D}_ℓ be the set of all infinite upper-triangular band matrices of the form

$$\begin{bmatrix} d_0(0) & \cdot & \cdot & \cdot & \cdot & d_\ell(0) & & 0 \\ d_0(1) & \cdot & \cdot & \cdot & \cdot & d_\ell(1) & & \\ \cdot & & & & & \cdot & \cdot & \\ & & & & & & \cdot & \\ & & & & & d_0(r) & \cdot & \cdot & \cdot & d_\ell(r) \\ 0 & & & & & \cdot & \cdot & & \cdot & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & \cdot \end{bmatrix}$$

where $d_0, d_1, \dots, d_\ell \in S$ and $d_0(r) \neq 0, d_\ell(r) \neq 0$ for infinitely many values of r . We shall also use the concise notation $[d_0, d_1, \dots, d_\ell]$ to represent infinite matrices in the set \mathcal{D}_ℓ . If we regard the infinite sequences in S as infinite column vectors, then an infinite sequence $Dx \in S$ is defined for each $D \in \mathcal{D}_\ell$, $x \in S$ by ordinary matrix multiplication. In fact, the terms of Dx are given by

$$Dx(r) = \sum_{s=0}^{\ell} d_s(r)x(r+s), \quad r = 0, 1, 2, \dots.$$

Clearly, each $D \in \mathcal{D}_\ell$ is a linear operator on S under this definition. We shall say that \mathcal{D}_ℓ is the class of linear difference operators of order ℓ on S .

Note that \mathcal{D}_ℓ is defined for each integer $\ell \geq 0$. We pointed out in §1.1 that intermediate solutions of linear difference equations

can exist only if the order of the equation is at least two. But we are going to factor linear difference operators into a product of two lower-order operators. Therefore, operators of order one will arise. Furthermore, the formalism to be introduced will allow even the degenerate case of operators of order zero.

If d_0 and d_ℓ have no terms equal to zero, we shall say the difference operator $[d_0, d_1, \dots, d_\ell] \in \mathcal{D}_\ell$ is nonsingular; compare [14, §12.0]. The terms of d_0 and d_ℓ will be called the leading and trailing coefficients of $[d_0, d_1, \dots, d_\ell]$, respectively.

We shall use the notation x^i to indicate a sequence from S whose first term is $x(i)$. Thus

$$x^i = (x(i), x(i+1), \dots).$$

The point i will be called the initial point of the sequence. When we do not indicate a superscript it is to be understood that zero is the initial point of the sequence. Similarly, we shall use the notation $x^{i,n}$ to indicate the (finite) subsequence of x^i whose final term is $x(n)$. Thus

$$x^{i,n} = (x(i), x(i+1), \dots, x(n)).$$

The point n will be called the terminal point of the subsequence.

Similarly, for linear difference operators from \mathcal{D}_ℓ , we shall use the notation D^i or $[d_0^i, d_1^i, \dots, d_\ell^i]$ to indicate that i is the initial point of each of the $\ell + 1$ sequences involved in the definition of the operator. See Figure 1. When the superscript is omitted, it is understood that these initial points are all zero unless it is clear from the context (as in equation (1.2.1) below) what these initial points are.

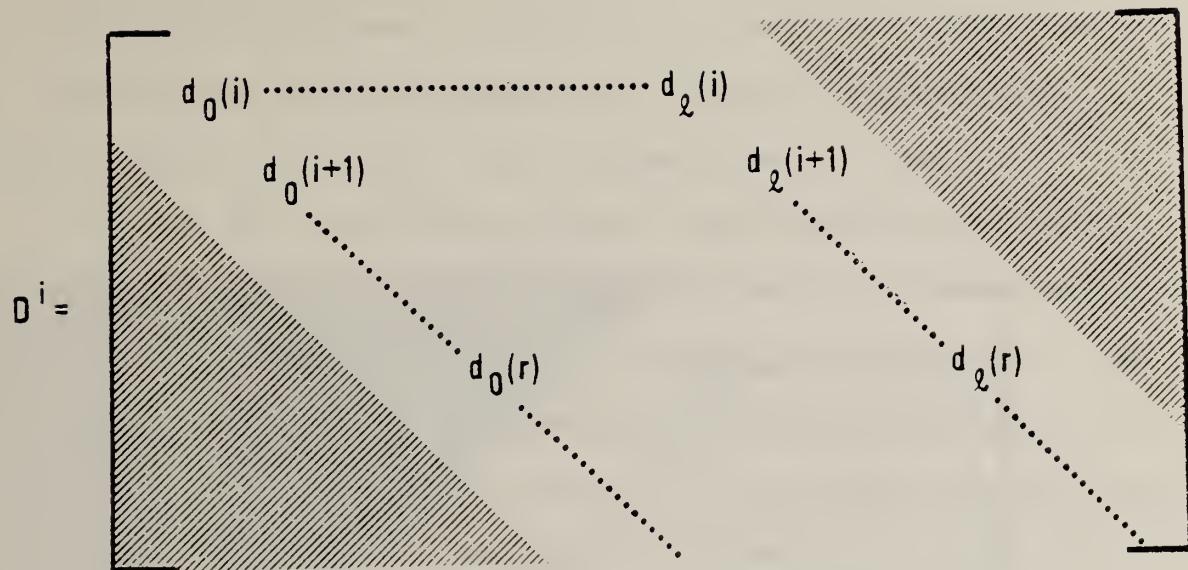


Figure 1. The operator D^i .

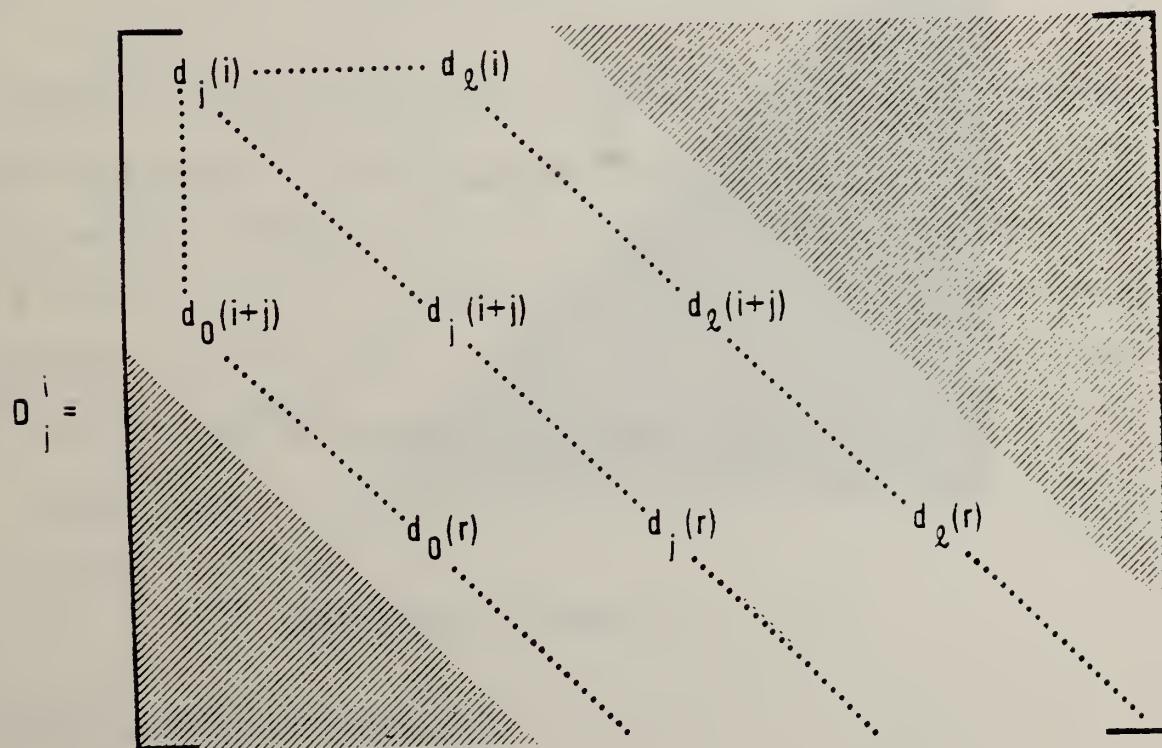


Figure 2. The matrix D_j^i .

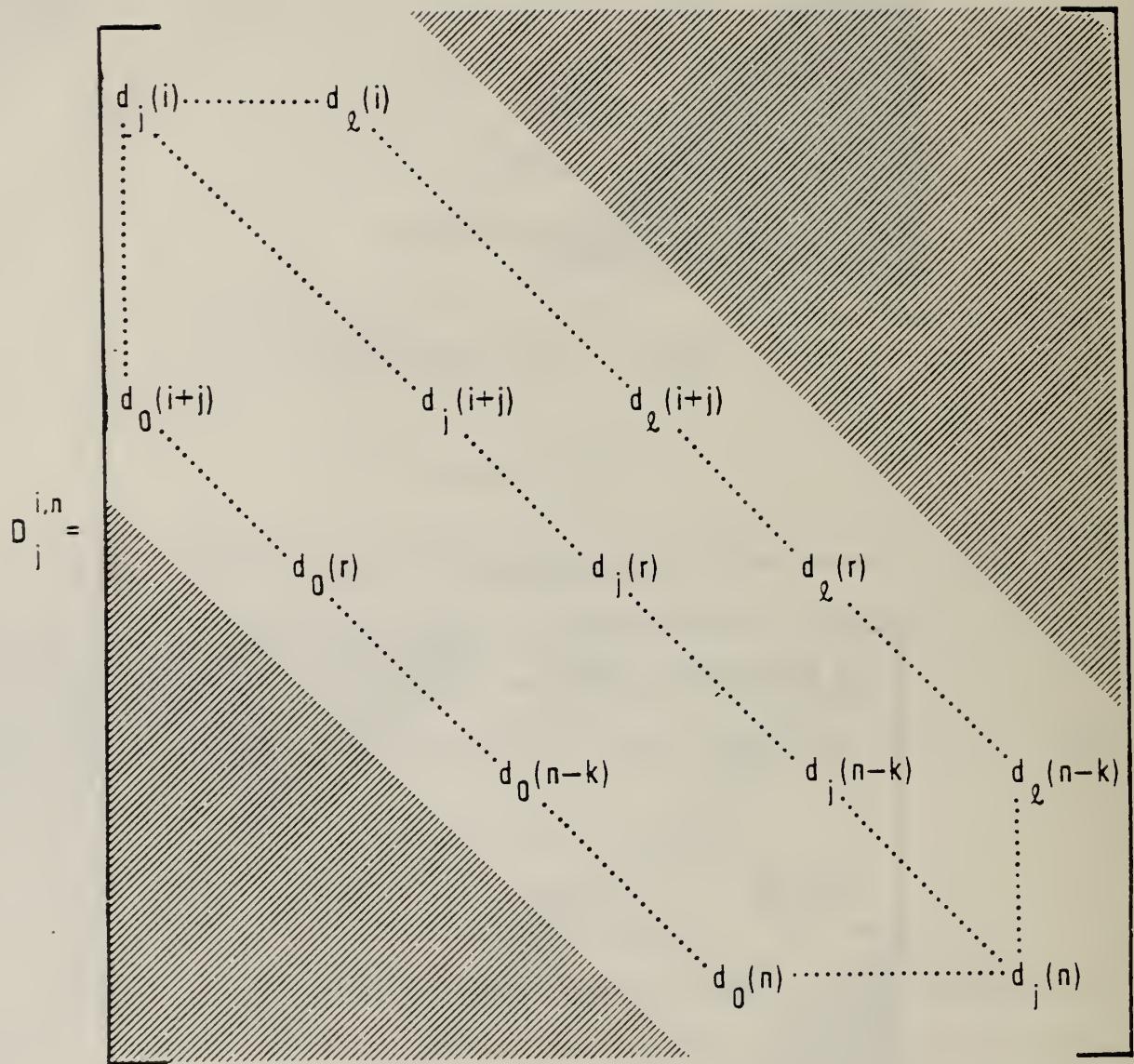


Figure 3. The matrix $D_j^{i,n}$.

Two further notational conventions to be used in this thesis are D_j^i , the infinite matrix obtained from D^i by deleting the first j columns ($0 \leq j \leq \ell$) ; and $D_j^{i,n}$, the leading principal submatrix of order $n - i + 1$ of D_j^i . See Figures 2 and 3. Note that each diagonal of D_j^i below the principal diagonal is a subsequence of one of the sequences $d_0^i, d_1^i, \dots, d_{j-1}^i$, the sequence of d_j^i is on the principal diagonal, and the sequences $d_{j+1}^i, \dots, d_\ell^i$ form the diagonals above the principal diagonal. The form of the finite matrix $D_j^{i,n}$ is similar, with the finite subsequence $d_j^{i,n}$ lying on the principal diagonal.

A linear difference equation is an equation of the form $Dx = g$, where $g \in S$ and $D \in \mathcal{D}_\ell$ are prescribed. Either $x \in S$ (or a subsequence of x of the form $x^{i,n}$) is to be determined. This equation is equivalent to the infinite linear algebraic system

$$Dx(r) = g(r), \quad r = 0, 1, \dots .$$

Analogously to linear differential equations, auxiliary conditions are required in order to specify a unique solution. In our investigations these auxiliary conditions will consist of two finite subsequences $x^{i,i+j-1}$ and $x^{n,n+k-1}$ (or possibly only one of these) which are to be specified in advance, where $j + k = \ell$ and $j \geq 0, k \geq 0$. These are called the initial and terminal conditions, respectively, when $n \geq i + j$.

Let us consider the following finite boundary value problem: Determine $y^{i+j,n-1}$ such that

$$(1.2.1) \quad Dy(r) = g(r), \quad r = i, i+1, \dots, n-j-1$$

together with initial conditions

$$(1.2.2) \quad y(i+r) = \alpha_r, \quad r = 0, 1, \dots, j-1$$

and terminal conditions

$$(1.2.3) \quad y(n+r) = \beta_r, \quad r = 0, 1, \dots, k-1$$

are satisfied, where $D \in \mathcal{D}_\ell$, $\ell \geq 1$, $g \in S$, $j \in \{0, 1, \dots, \ell\}$, $k = \ell - j$, $i \geq 0$, $n \geq i + j$ and $\alpha_0, \alpha_1, \dots, \alpha_{j-1}, \beta_0, \beta_1, \dots, \beta_{k-1} \in F$. In view of (1.2.2) and (1.2.3), the finite subsequences $y^{i, i+j-1}$ and $y^{n, n+k-1}$ are known. Of course, if $j = 0$ then $y^{i, i+j-1}$ is null. Similarly if $k = 0$. These cases correspond to the absence of either (1.2.2) or (1.2.3). The unknown subsequence $y^{i+j, n-1}$ satisfies a linear system of algebraic equations which can be expressed in matrix terms using the notation we introduced above. The system is

(1.2.4)

$$D_j^{i, n-j-1} y^{i+j, n-1} = g^{i, n-j-1} - \begin{bmatrix} D_0^{i, i+j-1} & 0 \\ \dots & \dots \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \dots \\ D_\ell^{n-\ell, n-j-1} y^{n, n+k-1} \end{bmatrix}$$

provided that $n - i - j \geq \max(j, k)$. The first partitioned column vector on the right side of (1.2.4) has j entries in the upper part, and the second partitioned column vector has k entries in the lower part. The length of the other column vectors appearing in (1.2.4) is $n - i - j$. This is what gives rise to the restriction $n - i - j \geq \max(j, k)$. This restriction is irrelevant for the applications we have in mind, since we shall be considering sequences of finite boundary value problems as $n \rightarrow \infty$. Note that the number of zeros represented by the symbol 0 in the partitioned column vectors is $n - i - 2j$ in the first vector and $n - i - \ell$ in the second vector. Obviously, the partitioned column vectors reflect the influence of the boundary conditions (1.2.2) and (1.2.3).

Solution of the boundary value problem (1.2.1) to (1.2.3) is equivalent to solution of the linear system (1.2.4). Equation (1.2.4) possesses

a unique solution if, and only if, the matrix $D_j^{i,n-j-1}$ is nonsingular.

If in addition we assume that the difference operator D is nonsingular, then the solution $y^{i,n-k-1}$ of (1.2.1) to (1.2.3) extends uniquely to $y \in S$ by forward and backward recurrence. However, nonsingularity of D is not necessary for this unique extension to exist; for example, some of the leading and trailing coefficients of D which appear in the finite matrix $D_j^{i,n-j-1}$ could be zero.

Let us consider the nonsingular triangular cases of (1.2.4). The first is given by $j = \ell$. Then (1.2.4) assumes the form

$$(1.2.4a) \quad D_\ell^{i,n-\ell-1} y^{i+\ell,n-1} = g^{i,n-\ell-1} - \begin{bmatrix} D_{\ell}^{i,i+\ell-1} y^{i,i+\ell-1} \\ 0 \\ \dots \\ 0 \end{bmatrix} .$$

Writing out these equations in detail, we have

$$\begin{aligned} d_\ell(i)y(i+\ell) &= g(i) - \sum_{s=0}^{\ell-1} d_s(i)y(i+s) \\ d_{\ell-1}(i+1)y(i+\ell) + d_\ell(i+1)y(i+\ell+1) &= g(i+1) - \sum_{s=0}^{\ell-2} d_s(i+1)y(i+s+1) \\ &\vdots \\ &\vdots \\ \sum_{s=0}^{\ell} d_s(i+\ell)y(i+\ell+s) &= g(i+\ell) \\ &\vdots \\ &\vdots \\ \sum_{s=0}^{\ell} d_s(n-\ell-1)y(n-\ell-1+s) &= g(n-\ell-1) \end{aligned}$$

Since this system is lower triangular, it is solvable by determining $y(i+\ell)$ from the first equation, $y(i+\ell+1)$ from the second equation, and so on. This process is equivalent to developing $y(i+\ell), y(i+\ell+1), \dots, y(n-1)$ from (1.2.1) by forward recurrence starting with the given values (1.2.2).

The second triangular case is given by $j = 0$. Then (1.2.4) assumes the form

$$D_0^{i,n-1} y^{i,n-1} = g^{i,n-1} - \begin{bmatrix} \dots & 0 \\ D_\ell^{n-\ell,n-1} y^{n,n+\ell-1} \end{bmatrix}$$

Writing out these equations in detail, we have

$$\begin{aligned} \sum_{s=0}^{\ell} d_s(i)y(i+s) &= g(i) \\ &\cdot \\ &\cdot \\ &\cdot \\ \sum_{s=0}^{\ell} d_s(n-\ell-1)y(n-\ell-1+s) &= g(n-\ell-1) \\ &\cdot \\ &\cdot \\ &\cdot \\ d_0(n-2)y(n-2) + d_1(n-2)y(n-1) &= g(n-2) - \sum_{s=2}^{\ell} d_s(n-2)y(n-2+s) \\ d_0(n-1)y(n-1) &= g(n-1) - \sum_{s=1}^{\ell} d_s(n-1)y(n-1+s) \end{aligned}$$

Since this system is upper triangular, it is solvable by determining $y(n-1)$ from the last equation, $y(n-2)$ from the next-to-last equation, and so on. This process is equivalent to developing $y(n-1), y(n-2), \dots, y(i)$ from (1.2.1) by backward recurrence starting with the values (1.2.3).

The nontriangular cases of (1.2.4) in which $0 < j < \ell$ will be of greater interest to us. Our primary goal is to find a way to solve these problems by performing, in effect, a forward recurrence of order j followed by a backward recurrence of order $\ell - j$.

Definition 1.2.1 A nonsingular boundary value problem (1.2.4), or equivalently (1.2.1) to (1.2.3), having initial point i , terminal point n , number of initial conditions j and difference operator $D = [d_0, d_1, \dots, d_\ell] \in \mathcal{D}_\ell$, will be called factorizable provided there

exist difference operators $A^i = [a_0^i, a_1^i, \dots, a_j^i]$ of order j and $B^{i+j} = [b_0^{i+j}, b_1^{i+j}, \dots, b_k^{i+j}]$ of order $k = \ell - j$ such that

$$(1.2.5) \quad D_j^{i,n-j-1} = A_j^{i,n-j-1} B_0^{i+j,n-1}.$$

If in addition the sequences a_j^i and b_0^{i+j} are free of zeros and the infinite matrix factorization

$$(1.2.6) \quad D_j^i = A_j^i B_j^{i+j}$$

is valid, then we shall say that the difference operator D is (i,j) - factorizable.

Several remarks may be made here. Note that $D_j^{i,n-j-1}$ and D_j^i are band matrices each having total bandwidth $\ell + 1$, lower bandwidth j , and upper bandwidth k . A nonsingular and factorizable boundary value problem is one for which a finite matrix factorization of $D_j^{i,n-j-1}$ exists such that the left factor is lower triangular with total bandwidth $j + 1$ and the right factor is upper triangular with total bandwidth $k + 1$. The difference operators A^i and B^{i+j} may be arbitrary as long as they satisfy the condition (1.2.5), since (1.2.5) involves only a finite submatrix of each of A^i and B^{i+j} . Equation (1.2.5) is illustrated in Figure 4. Since $D_j^{i,n-j-1}$ is nonsingular, both $A_j^{i,n-j-1}$ and $B_0^{i+j,n-1}$ are nonsingular. Thus the entries on the principal diagonals of $A_j^{i,n-j-1}$ and $B_0^{i+j,n-1}$ are all nonzero.

Now suppose the difference operator $D \in \mathcal{D}_\ell$ is (i,j) - factorizable; it need not be a nonsingular operator. Because of the triangular forms of A_j^i and B^{i+j} , equation (1.2.6) implies equation (1.2.5) for every $n \geq i + j + 1$. In words, every leading principal submatrix of D_j^i is equal to the product of the corresponding leading principal submatrices of A_j^i and B^{i+j} . The assumption that the subsequences a_j^i and b_0^{i+j} are free of zeros is equivalent to assuming that the entries on the

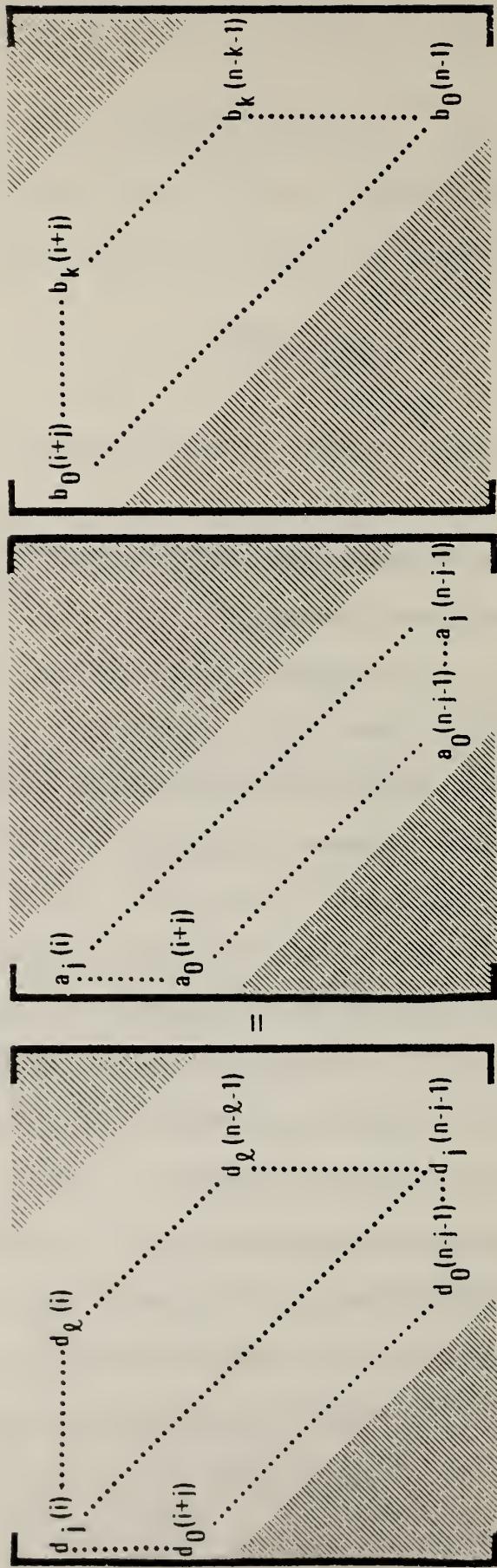


Figure 4. The factorization (1.2.5).

principal diagonals of the infinite matrices A_j^i and B^{i+j} are all non-zero. It follows that every leading principal submatrix of D_j^i is nonsingular. Thus we have proved the following theorem:

Theorem 1.2.1 Let $D \in \mathcal{D}_\ell$ be (i,j) - factorizable. Then, for every $n \geq i + j + \max(j, \ell-j)$, the boundary value problem (1.2.4) having difference operator D , initial point i , number of initial conditions j and terminal point n is nonsingular and factorizable. \blacksquare

The next theorem shows it is possible to obtain a solution to a factorizable boundary value problem, in theory at least, by performing an appropriate forward recurrence followed by an appropriate backward recurrence.

Theorem 1.2.2 Let (1.2.4) be a nonsingular and factorizable boundary value problem with initial point i , number of initial conditions j , terminal point n and difference operator $D \in \mathcal{D}_\ell$. If A^i and B^{i+j} are difference operators of order j and $k = \ell-j$, respectively, such that (1.2.5) is valid, then the solution of (1.2.4) is identical to the solution of

(1.2.7)

$$B_0^{i+j, n-1} y^{i+j, n-1} = z^{i+j, n-1} - \begin{bmatrix} & & & & 0 \\ & \dots & & & \dots \\ & & & & 0 \\ (A_j^{n-\ell, n-j-1})^{-1} D_\ell^{n-\ell, n-j-1} y^{n, n+k-1} & & & & \end{bmatrix}$$

where $z^{i+j, n-1}$ is the solution of

(1.2.8)

$$A_j^{i, n-j-1} z^{i+j, n-1} = g^{i, n-j-1} - \begin{bmatrix} D_0^{i, i+j-1} y^{i, i+j-1} \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

Proof The matrices $A_j^{i,n-j-1}$ and $B_0^{i+j,n-1}$ are nonsingular because $D_j^{i,n-j-1}$ is nonsingular; see (1.2.5). Therefore, the diagonal elements of $A_j^{i,n-j-1}$ and $B_0^{i+j,n-1}$ (which are triangular matrices) are nonzero and (1.2.7) and (1.2.8) are nonsingular linear systems. Furthermore, $A_j^{n-\ell,n-j-1}$ is nonsingular because it is a principal submatrix of $A_j^{i,n-j-1}$. Therefore the right side of (1.2.7) is unambiguously defined.

Now, premultiplying (1.2.7) by $A_j^{i,n-j-1}$ and using (1.2.5) and (1.2.8), we obtain

$$D_j^{i,n-j-1} y^{i+j,n-1} = g^{i,n-j-1} - \begin{bmatrix} D_0^{i,i+j-1} y^{i,i+j-1} \\ \dots \\ 0 \end{bmatrix} - A_j^{i,n-j-1} \begin{bmatrix} \dots & 0 \\ (A_j^{n-\ell,n-j-1})^{-1} D_\ell^{n-\ell,n-j-1} y^{n,n+k-1} \end{bmatrix}.$$

Consider the last product on the right side of this equation. Its block structure is

$$\begin{bmatrix} A_j^{i,n-\ell-1} \\ \dots \\ * \end{bmatrix} \begin{bmatrix} \dots & 0 \\ (A_j^{n-\ell,n-j-1})^{-1} D_\ell^{n-\ell,n-j-1} y^{n,n+k-1} \end{bmatrix}$$

where 0 in the matrix denotes a zero submatrix with $n - i - \ell$ rows and $k (= \ell - j)$ columns, 0 in the vector denotes a zero subvector of length $n - i - \ell$, and * in the matrix denotes a submatrix which needs no further identification here. Therefore, the product in question is equal to the vector

$$\begin{bmatrix} \dots & 0 \\ D_\ell^{n-\ell,n-j-1} y^{n,n+k-1} \end{bmatrix}.$$

Substitution into the equation and comparison with (1.2.4) completes the proof. \blacksquare

An important remark is that if (1.2.6) is valid, i.e. if D is (i,j) -factorizable with left factor A^i and right factor B^{i+j} , then

$$D_{\ell}^{n-\ell, n-j-1} = A_j^{n-\ell, n-j-1} B_k^{n-k, n-1}.$$

In this case $B_k^{n-k, n-1}$ may be written in place of $(A^{n-\ell, n-j-1})^{-1} D^{n-\ell, n-j-1}$ in (1.2.7). The truth of this remark is apparent from Figure 4.

The next theorem gives necessary and sufficient conditions for a difference operator to be (i,j) - factorizable. The proof also shows that (i,j) - factorizations may be constructed by Gaussian elimination in natural order, i.e., without any row or column interchanges.

Theorem 1.2.3 Let $D \in \mathcal{D}_{\ell}$. Then

i) A nonsingular boundary value problem having initial point i , number of initial conditions j and terminal point n is factorizable if every leading principal submatrix of $D_j^{i, n-j-1}$ is nonsingular.

ii) D is (i,j) - factorizable if every leading principal submatrix of D_j^i is nonsingular. Furthermore, this condition is necessary if D is nonsingular.

Proof Our approach is to apply Gaussian elimination [10, §2.1] without any row or column interchanges. Because Gaussian elimination is so familiar, we merely sketch the proof. The proof of (i) is contained in the first $n - i - j$ stages of the proof of (ii); therefore we supply only the proof of (ii). We begin by proving sufficiency of the condition.

The first row of D_j^i is used to annihilate every nonzero element below the first element in the first column. This is possible because the pivot element, which is the leading principal minor of order one, is nonzero. Because of the band structure, only the j rows immediately follow-

ing the first row are affected. Also (again because of the band structure) the trailing coefficient in each of these rows is not changed. The multipliers used in these annihilations go into positions 2 through $j + 1$ of the first column of the lower triangular factor A_j^i . The first element is unity and every other element of the first column of A_j^i is zero. This completes the first stage.

Now suppose the $(r-1)$ st stage has been completed. Let the notation \bar{D}_j^i denote the transformation of the original D_j^i resulting from the first $r - 1$ stages. The leading principal submatrix of \bar{D}_j^i of order r is upper triangular, having resulted in effect from the corresponding principal submatrix of the original D_j^i by Gaussian elimination in natural order. Its diagonal elements are all nonzero, by hypothesis. Therefore the pivotal element is nonzero and the r -th row of \bar{D}_j^i may be used to annihilate every nonzero element below the r -th element in the r -th column. Only the j rows immediately following the r -th row are affected, and the trailing coefficient in each of these rows is not changed. The multipliers used in this annihilation go into positions $r + 1$ through $r + j$ of the r -th column of A_j^i ; unity goes into the r -th position, and zeros go into every other position. This completes the r -th stage.

Let B^{i+j} be the infinite upper triangular matrix which results from the full sequence of transformations on D_j^i . By our construction, both A_j^i and B^{i+j} have the bandwidths required by (i,j) - factorizability. Also by our construction, formally multiplying A_j^i on the right by B^{i+j} results in D_j^i ; compare (1.2.6).

In order to prove necessity of the condition in part (ii), note that the factorization produced by the foregoing process is such that leading and trailing coefficients of both A_j^i and B^{i+j} are nonzero if D is

nonsingular. Therefore, no row or column interchanges could be made without widening the bandwidth. Therefore, Gaussian elimination in natural order is the only possibility for arriving at the factorization. But Gaussian elimination in natural order is possible only if every leading principal minor is nonzero. This completes the proof. \square

Examination of the proof of Theorem 1.2.3 shows that a unique (i,j) -factorization does not exist, since a different scaling of the rows is possible. This would result in the appearance of entries on the principal diagonal of A_j^i that are other than unity. In fact, the scaling used in the $(0,1)$ -factorization in Olver's second-order algorithm put the sequence $(p(2), p(3), \dots)$ on the principal diagonal of the right factor; compare §1.1. Thus the principal diagonal of the left factor will contain entries other than unity, in general.

1.3 A Preliminary Development of the Algorithm

Let $D \in \mathcal{D}_\ell$. It is clear that an extension of Olver's algorithm for computing a particular solution y of a given difference equation $Dy = g$ should involve the posing of a boundary value problem whose solution y_n is an approximation of y over some finite subsequence $y^{i,m}$. The boundary value problem will have some number $j \geq 0$ of initial conditions (depending on the particular solution being sought). These will consist of a subsequence of j consecutive values of y . The terminal conditions of the boundary value problem will be $y_n^{(n)} = y_n^{(n+1)} = \dots = y_n^{(n+k-1)} = 0$, where $k = \ell - j$.

The first task is to decide how to choose j . A necessary prerequisite is a classification of possible solution types when the operator is of order higher than two. Oliver [19] provided an early extension of Olver's algorithm. He also provided the following definition of dominance. Suppose we are given two sequences $a, b \in S$. Then b dominates a if there exists some $i \geq 0$ such that

$$(1.3.1) \quad |b(r+1)/b(r)| > |a(r+1)/a(r)|$$

for all $r \geq i$. Oliver considered only those difference operators for which a system of complementary solutions could be found such that the complementary solutions are linearly ordered under this definition of dominance. Furthermore, he considered only those particular solutions of the given linear difference equation which could be compared to each of the complementary solutions. But not all possible particular solutions can be so compared; for example, no particular solution which has an infinite subsequence of zeros can play the role of either a or b in (1.3.1).

In §1.1, we adopted a different definition of dominance: b dominates a if

$$(1.3.2) \quad \lim_{r \rightarrow \infty} \frac{a(r)}{b(r)} = 0 .$$

This differs from Oliver's definition (which is equivalent to requiring that ultimately $|a(r)/b(r)|$ be monotonically decreasing). Another difference is that the limit of $|a(r)/b(r)|$ under Oliver's definition may be greater than zero, as the example $a(r) = 1 + 1/r$ and $b(r) = 1$ immediately shows.

Let $D \in \mathcal{D}_\ell$. The set $K(D) = \{x \in S \mid Dx=0\}$ will be called the kernel of D . Obviously, $K(D)$ is a linear subspace of S . Let us suppose that $K(D)$ has dimension ℓ , so that there exist ℓ linearly independent sequences $x_1, x_2, \dots, x_\ell \in K(D)$ which span $K(D)$. A sufficient condition for this to be the case is that D be nonsingular; compare [14, §12.1]. We shall say that the set x_1, x_2, \dots, x_ℓ forms a basis for D .

Definition 1.3.1 We shall say that $D \in \mathcal{D}_\ell$ is totally separable (as $r \rightarrow \infty$) provided there exists a basis y_1, y_2, \dots, y_ℓ for D such that y_{s+1} dominates y_s in the sense of (1.3.2) for each s , $1 \leq s < \ell$.

Any basis which satisfies the conditions of Definition 1.3.1 will be called totally ranked. Such a basis is not unique because y_1 , for example, could be added to each of y_2, \dots, y_ℓ and the resulting basis would also be totally ranked.

For each s , $1 \leq s \leq \ell$, let K_s denote the linear subspace of S which is spanned by y_1, y_2, \dots, y_s . In particular, $K_\ell = K(D)$. It seems obvious, and it will be proved in a more general setting in Chapter 2, that every totally ranked basis generates the same subspaces. The subspaces satisfy

$$K_1 \subset K_2 \subset \dots \subset K_\ell = K(D) .$$

Each set inclusion is proper, since the dimension of K_s is s . This ascending chain of subspaces induces an obvious classification of the solutions of $Dy = 0$ into ℓ distinct types. However, we are interested in classifying the solutions of $Dy = g$ for nonzero, and indeed, arbitrary $g \in S$. Therefore, we frame the definition so as to cover all cases:

Definition 1.3.2 Let $D \in \mathcal{D}_\ell$ be totally separable with totally ranked basis y_1, y_2, \dots, y_ℓ . Then, for each $y \in S$, we shall say that the type of y (with respect to D) is s , where $0 < s < \ell$, and we shall write $\text{type}(y) = s$, provided that

$$(1.3.3) \quad \lim_{r \rightarrow \infty} \frac{y(r)}{y_{s+1}(r)} = 0$$

and

$$(1.3.4) \quad \infty \geq \limsup_{r \rightarrow \infty} \left| \frac{y(r)}{y_s(r)} \right| > 0 .$$

Furthermore, if (1.3.3) is true for $s = 0$, we define $\text{type}(y) = 0$, and if (1.3.4) is true for $s = \ell$, we define $\text{type}(y) = \ell$.

The type of each $y \in S$ is the same regardless of which totally ranked basis is used. This will be proved for a more general classification of sequences in Chapter 2. Thus we are justified in defining the type with respect to D rather than with respect to a particular totally ranked basis.

The conditions for Olver's algorithm [21] suggest that Definitions 1.3.1 and 1.3.2 are appropriate for an extension of the algorithm to arbitrary order; see also Cash [6]. But it would be desirable to have a classification scheme which is applicable, for example, to every difference

equation with constant coefficients. Not every constant-coefficient operator is totally separable, however. In the next chapter we shall introduce a more general class of difference operators (separable operators) which does include every constant-coefficient operator.

Incidentally, the class of difference equations considered by Oliver [19] includes every constant-coefficient equation, since the standard basis (see §2.1 below) is linearly ordered under his definition of dominance.

Before stating the next theorem, we explain what we mean by matrices and determinants associated with the name Casorati. If $x_1, x_2, \dots, x_s \in S$ are arbitrary sequences, then for each $r \geq 0$ the $s \times s$ matrix

$$\begin{bmatrix} x_1(r) & x_2(r) & \dots & x_s(r) \\ x_1(r+1) & x_2(r+1) & \dots & x_s(r+1) \\ \vdots & \vdots & & \vdots \\ x_1(r+s-1) & x_2(r+s-1) & \dots & x_s(r+s-1) \end{bmatrix}$$

will be called a Casorati matrix. Let $X = \{x_1, x_2, \dots, x_s\}$. We shall use the concise notation $[X](r)$ to denote the Casorati matrix above. Furthermore, $|X|(r)$ will denote the Casoratian (of X at r), defined by

$$|X|(r) = \det [X](r), \quad r = 0, 1, \dots.$$

If $D \in \mathcal{D}_\ell$ is nonsingular, Casorati's theorem [14, §12.11] states that x_1, x_2, \dots, x_ℓ forms a basis for D if, and only if, $|X|(r) \neq 0$ for all r .

Theorem 1.3.1 Let y be a solution of the difference equation $Dy = g$, where $D \in \mathcal{D}_\ell$ has a totally ranked basis $Y = \{y_1, y_2, \dots, y_\ell\}$. Let

$j = \text{type}(y)$. Then y is uniquely determined by its values $y(i), y(i+1), \dots, y(i+j-1)$ provided that $i \geq 0$ is a point at which the leading principal minor of order j of $|Y|(i)$ is nonzero.

Proof Assume $j = 0$; then the condition on the Casoratian is vacuous and we must show y is uniquely determined without knowledge of any of its values. Suppose $z \neq y$ is another solution of type zero; then there exist unique scalars $\alpha_1, \alpha_2, \dots, \alpha_\ell$ (not all zero) such that

$$y(r) = z(r) + \alpha_1 y_1(r) + \dots + \alpha_\ell y_\ell(r), \quad r = 0, 1, \dots.$$

Let α_t be the nonzero scalar of highest subscript. Dividing through by $y_t(r)$ and allowing $r \rightarrow \infty$, we see that $\alpha_t = 0$, which is a contradiction.

Next, assume $j > 0$. Suppose $z \neq y$ is another solution of type j such that

$$z(i+r) = y(i+r), \quad r = 0, 1, \dots, j-1.$$

There exist unique scalars $\alpha_1, \alpha_2, \dots, \alpha_\ell \in F$ (not all zero) such that

$$y(r) = z(r) + \alpha_1 y_1(r) + \dots + \alpha_\ell y_\ell(r), \quad r = 0, 1, \dots.$$

It is proved easily that each of $\alpha_{j+1}, \dots, \alpha_\ell$ is zero by a method similar to that of the previous paragraph. The remaining scalars are determined by the linear system

$$\alpha_1 y_1^{i, i+j-1} + \dots + \alpha_j y_j^{i, i+j-1} = 0$$

(where 0 stands for a column vector of j zeros and we have used the notation developed in §1.2). This system is nonsingular by the assumption on the Casoratian; therefore, $\alpha_t = 0$, $t = 1, 2, \dots, j$, which is a contradiction. \square

Theorem 1.3.1 confirms the adequacy of the selection of j initial conditions in the approximating boundary value problem, where $j = \text{type}(y)$. In §3.3 we will demonstrate that stability considerations demand exactly j initial conditions, in general, for a solution of type j .

Now let us suppose that a value of j has been selected, where $j \geq \text{type}(y)$. Since $y_n^{i,i+j-1} = y^{i,i+j-1}$ and $y_n^{n,n+k-1} = 0$, the linear system to be solved is

$$(1.3.5) \quad D_j^{i,n-j-1} y_n^{i+j,n-1} = g^{i,n-j-1} - \begin{bmatrix} D_0^{i,i+j-1} y^{i,i+j-1} \\ \dots \\ 0 \end{bmatrix};$$

compare (1.2.4). The value of n is to be found in the course of the computation. In view of Theorem 1.3.1, y is the unique solution of the infinite linear system

$$(1.3.6) \quad D_j^i y^{i+j} = g^i - \begin{bmatrix} D_0^{i,i+j-1} y^{i,i+j-1} \\ \dots \\ 0 \end{bmatrix}.$$

We assume that D is (i,j) -factorizable, i.e., that linear difference operators $A_j^i \in \mathcal{D}_j$ and $B_{\ell-j}^{i+j} \in \mathcal{D}_{\ell-j}$ exist such that $D_j^i = A_j^i B_{\ell-j}^{i+j}$.

As shown in the proof of Theorem 1.2.3, successive rows of this factorization, and indeed of the triangularization of the infinite system (1.3.6), may be generated indefinitely by the forward stage of Gaussian elimination without pivoting. The stability of this process, also, will be discussed in §3.3.

In consequence of Theorems 1.2.1 and 1.2.2 and the terminal conditions $y_n^{n,n+k-1} = 0$, the solution of (1.3.5) may be obtained as the solution of

$$(1.3.7) \quad B_0^{i+j,n-1} y_n^{i+j,n-1} = z^{i+j,n-1}$$

where $z^{i+j, n-1}$ is a finite subsequence of the solution of

$$(1.3.8) \quad A_j^i z^{i+j} = g^i - \begin{bmatrix} D_{0, i+j-1}^{i, i+j-1} & y \\ \dots & \dots \\ 0 & 0 \end{bmatrix} .$$

Comparison of (1.3.8) and (1.3.6) and reference to the identity

$$D_j^i = A_j^i B^{i+j} \text{ show that}$$

$$(1.3.9) \quad B^{i+j} y^{i+j} = z^{i+j} .$$

Thus (1.3.7) represents the finite linear system that results from (1.3.9) by truncation at the $(n-i-j)$ th row and column. Since B^{i+j} and z^{i+j} are generated by Gaussian elimination without pivoting, the solution of (1.3.5) for any n is easily programmed for an automatic digital computer.

The algorithm comprises two stages: (i) forward elimination (solution of (1.3.8)); (ii) back substitution (solution of (1.3.7)). The forward elimination can be continued arbitrarily far. The back substitution is numerically equivalent to backward recurrence, and proceeds until we return to the initial point i .

The question of the stability of the back substitution is easily settled. For totally separable operators we have the following theorem:

Theorem 1.3.2 Let $D \in \mathcal{D}_\ell$ have a totally ranked basis $Y = \{y_1, y_2, \dots, y_\ell\}$. Suppose $i \geq 0$ and $j \in \{0, 1, \dots, \ell-1\}$ are such that D is (i, j) -factorizable and the leading principal minor of order j of $|Y|(i)$ is nonzero. If $D_j^i = A_j^i B^{i+j}$ is an (i, j) -factorization of D , then B^{i+j} is totally separable. Furthermore, if $z \in S$ has type j or less with respect to D , then z^{i+j} has type zero with respect to B^{i+j} .

Proof Suppose $j = 0$. Then A_j^i is a diagonal matrix with nonzero elements on the diagonal; see Definition 1.2.1. For each $y \in Y$ we have

$$D^i y^i = A^i B^i y^i = 0, \text{ i.e.}$$

$$\sum_{s=0}^{\ell} a_0(r) b_s(r) y(r+s) = 0, \quad r = i, i+1, \dots.$$

Therefore,

$$\sum_{s=0}^{\ell} b_s(r) y(r+s) = 0, \quad r = i, i+1, \dots$$

which proves $\{y^i | y \in Y\}$ is a totally ranked basis for B^i . It follows that z^i has type zero with respect to B^i whenever z has type zero with respect to D .

Suppose $0 < j < \ell$. Let $U = \{y_1, y_2, \dots, y_j\}$. Since $|U|(i) \neq 0$ it is possible to find a linear combination x of y_1, y_2, \dots, y_j such that $x(i), x(i+1), \dots, x(i+j-1)$ take on any prescribed set of values. Therefore, appropriate linear combinations x_s , $s = j+1, j+2, \dots, \ell$, can be found such that $y_s(r) + x_s(r)$ vanish for $r = i, i+1, \dots, i+j-1$. Let $v_{s-j} = y_s + x_s$, $s = j+1, j+2, \dots, \ell$. It is easy to verify that $\{y_1, \dots, y_j, v_1, \dots, v_k\}$ is a totally ranked basis for D . Now let $v \in \{v_1, v_2, \dots, v_k\}$. In view of Theorems 1.2.1 and 1.2.2, the remark following Theorem 1.2.2, and the facts that $Dv = 0$ and $v(i+r) = 0$ for $r = 0, 1, \dots, j-1$, we conclude that $v^{i+j, n-1}$ satisfies

$$B_0^{i+j, n-1} v^{i+j, n-1} = - \begin{bmatrix} & & 0 \\ & \dots & \dots \\ B_k^{n-k, n-1} & v^{n, n+k-1} \end{bmatrix}$$

for every $n \geq i + j + \max(j, k)$; compare equation (1.2.7). On letting $n \rightarrow \infty$ this proves $B^{i+j} v^{i+j} = 0$. Thus $\{v_1^{i+j}, v_2^{i+j}, \dots, v_k^{i+j}\}$ is a totally ranked basis for B^{i+j} . Finally, let z be of type j or less with respect to D . Let $s \in \{1, 2, \dots, k\}$. Then

$$\frac{z(r)}{v_s(r)} = \frac{z(r)}{y_{j+s}(r)\{1+o(1)\}} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

which shows that z^{i+j} has type zero with respect to B^{i+j} . \square

In consequence of Theorem 1.3.2 and our remarks in §1.1, when $j \geq \text{type}(y)$ the back substitution stage of the algorithm is stable, at least for all r in a sufficiently large range.

It remains to determine the appropriate value of the truncation parameter n . It is necessary to introduce Green's formula for linear difference operators; see, e.g., Miller [13]. We apply Green's formula not to the operator D but to the operator B^{i+j} . Our development is essentially the same as that of Cash [6].

Let $B = [b_0, b_1, \dots, b_k] \in \mathcal{D}_k$. The infinite upper triangular matrix $\hat{B} = [\hat{b}_0, \hat{b}_1, \dots, \hat{b}_n] \in \mathcal{D}_k$ which is obtained from B by deleting the first k columns and transposing the result, i.e. whose entries are defined by

$$(1.3.10) \quad \hat{b}_s(r) = b_{k-s}(r+s),$$

is a linear difference operator of order k . We shall call \hat{B} the operator which is adjoint to B , or more briefly, the adjoint operator of B . An operator and its adjoint satisfy a number of elementary properties which are easily verified from the definition. Among these are

$$(1.3.11) \quad \hat{B}^P = (B_k^P)^{\text{Tr}}$$

$$(1.3.12) \quad \hat{B}_0^{P,Q} = (B_k^{P,Q})^{\text{Tr}}$$

$$(1.3.13) \quad \hat{B}_k^P = (B^{P+k})^{\text{Tr}}$$

$$(1.3.14) \quad \hat{B}_k^{p,q} = (B_0^{p+k, q+k})^{\text{Tr}} ,$$

valid for $q \geq p$ and $p = 0, 1, 2, \dots$.

Now take any $u, v \in S$ and $B \in \mathcal{D}_k$, and assume $k \geq 1$. Green's formula is the identity

$$(1.3.15) \quad \sum_{r=p}^{q-1} \{v(r)Bu(r) - u(r+k)\hat{B}v(r)\}$$

$$= (v^{p,p+k-1})^{\text{Tr}} B^{p,p+k-1} u^{p,p+k-1}$$

$$- (v^{q,q+k-1})^{\text{Tr}} B^{q,q+k-1} u^{q,q+k-1}$$

valid for $q \geq p + k$, $p \geq 0$. Green's formula may be verified by expressing the left side of (1.3.15) in the form

$$(v^{p,q-1})^{\text{Tr}} (Bu)^{p,q-1} - (u^{p+k,q+k-1})^{\text{Tr}} (\hat{B}v)^{p,q-1}$$

and reducing this expression with the aid of the definition (1.3.10) and the identities (1.3.11) - (1.3.14).

Theorem 1.3.3 Let $D_j^i = A_j^i B^{i+j}$ be an (i,j) -factorization of $D \in \mathcal{D}_\ell$, where D is totally separable and $j < \ell$. Let y be a solution of $Dy = g$, where $\text{type}(y) \leq j$ and $g \in S$. Let $y_n^{i+j,n-1}$ and z^{i+j} be the corresponding solutions of (1.3.5) and (1.3.8). Also, let s be a fixed integer satisfying $i + j + k - 1 \leq s < n$ and $w_s^{s+1} = (w_s(s+1), w_s(s+2), \dots)$ be the solution of

$$(1.3.16) \quad \hat{B}_k^{s-k+1} w_s^{s+1} = - \begin{bmatrix} \hat{B}^{s-k+1,s} & w_s^{s-k+1,s} \\ 0 & \dots \\ \dots & 0 \end{bmatrix}$$

where $k = \ell - j$ and

$$(1.3.17) \quad w_s^{s-k+1, s} = (0, 0, \dots, 0, 1) .$$

Then

$$(1.3.18) \quad y_n(s) = \sum_{r=s}^{n-1} w_s(r) z(r) / b_0(s) ,$$

where $b_0(s)$ is the $(s-i-j+1)$ st leading coefficient of B^{i+j} .

Remark: Equations (1.3.16), (1.3.17) are equivalent to the initial value problem

$$\sum_{t=0}^k \hat{b}_t(r) w_s(r+t) = 0 , \quad r = s-k+1, s-k+2, \dots$$

with initial conditions $w_s(s-k+1) = \dots = w_s(s-1) = 0$, $w_s(s) = 1$; compare (1.2.1)-(1.2.3) and (1.2.4a).

Proof: Let $p = s - k + 1$, $q = n$, $v = w_s$ and $u = y_n$ in (1.3.15). By virtue of (1.3.7), (1.3.16) and (1.3.17), the left side of (1.3.15) becomes

$$\sum_{r=s}^{n-1} w_s(r) z(r) .$$

Since $u^{q, q+k-1} = y_n^{n, n+k-1} = 0$, the second term on the right side of (1.3.15) is zero. Thus, expanding the right side and using (1.3.17), we derive

$$(v^{p,p+k-1}) \operatorname{Tr}_{B^{p,p+k-1}} u^{p,p+k-1}$$

$$= (0, 0, \dots, 0, 1) \begin{bmatrix} b_0(s-k+1) & \dots & b_{k-1}(s-k+1) \\ \vdots & \ddots & \vdots \\ & \ddots & \vdots \\ & & b_0(s) \end{bmatrix} \begin{bmatrix} y_n(s-k+1) \\ \vdots \\ y_n(s) \end{bmatrix}$$

$$= b_0(s) y_n(s) .$$

Equating the left and right sides thus arrived at, we obtain (1.3.18). \blacksquare

In §3.2 with appropriate conditions it will be proved that

$$\lim_{n \rightarrow \infty} y_n(r) = y(r)$$

for each fixed $r \geq i + j$. Thus, allowing $n \rightarrow \infty$ in (1.3.18), we have

$$(1.3.19) \quad y(s) = \sum_{r=s}^{\infty} w_s(r) z(r) / b_0(s)$$

for each $s \geq i + j + k - 1$. Similarly, if

$$\eta_n(s) = y(s) - y_n(s)$$

is the truncation error incurred in accepting the approximation $y_n(s)$

for $y(s)$, we have

$$(1.3.20) \quad \eta_n(s) = \sum_{r=n}^{\infty} w_s(r) z(r) / b_0(s)$$

for each $s \geq i + j + k - 1$.

Let us compare (1.3.20) with (1.1.2), the truncation error expansion for the original algorithm of Olver. Suppose $\ell = 2$ and $j = k = 1$.

The difference operator which is implicit in (1.1.1) is $D = [d_0, d_1, d_2]$ where

$$d_0(r) = a(r+1), \quad d_1(r) = -b(r+1), \quad d_2(r) = c(r+1)$$

for $r = 0, 1, 2, \dots$. Similarly, the right side of (1.1.1) is, in our present notation,

$$g(r) = d(r+1), \quad r = 0, 1, 2, \dots$$

The rows of the (0,1)-factorization $D_1^0 = A_1^0 B^1$ are scaled so that

$$b_0(r) = p(r+1), \quad b_1(r) = -p(r)$$

for $r = 1, 2, \dots$, where $B^1 = [b_0^1, b_1^1]$ and $p = (p(0), p(1), \dots)$ is the sequence defined in §1.1. The solution z^1 of the forward elimination stage of the algorithm is given by

$$z(r) = e(r), \quad r = 1, 2, \dots;$$

compare (1.3.8). Finally, it may be verified readily that the solution of the adjoint equation $\hat{B}w_s(r) = 0$, $r = s, s+1, \dots$ such that $w_s(s) = 1$ is given by

$$w_s(r) = \frac{p(s)p(s+1)}{p(r)p(r+1)}, \quad r = s, s+1, \dots.$$

Making these substitutions into (1.3.20) and exchanging the roles of r and s , we obtain (1.1.2).

The simple form of the equation for $w_s(r)$ depends solely on the particular scaling of the (0,1)-factorization used by Olver and the fact that $k = 1$. A similar simplification can be obtained for difference equations of arbitrary order provided only that $k = 1$. It stems from

the fact that the order of the adjoint equation is one, hence all solutions can be expressed as multiples of a single particular solution. These conclusions were also arrived at in Cash's paper [6].

The expansions (1.3.19) and (1.3.20) may be employed in the estimation of the optimal value of n such that a given termination criterion is met. Typical termination criteria would be to require one or more values of $y_n(r)$ in the range $i + j \leq r \leq m$ to approximate $y(r)$ to a specified absolute or relative precision. Since the back substitution stage is stable, controlling the absolute or relative error only at the point $r = m$ is often sufficient. In practice equations (1.3.19) and (1.3.20) are employed by replacing the upper limit by $n + v$, where $v \geq 1$ is a parameter chosen to suit the problem at hand.

The solution of (1.3.16), (1.3.17) is obtained by forward recurrence. The stability of this process will depend on the behavior of $w_s(r)$ for large r relative to the other solutions of the adjoint equation. In general there is no reason to expect that $w_s(r)$ will not contain a component of a maximal solution. Thus the solution of (1.3.16) will normally be stable, although there could be an initial range where some unstable rounding error propagation takes place.

CHAPTER 2. CLASSIFICATION OF SEQUENCES

2.1 Constant-Coefficient Operators

Let $D = [d_0, d_1, \dots, d_\ell] \in \mathcal{D}_\ell$ be a constant-coefficient operator. By this we mean that there exist $\delta_0, \delta_1, \dots, \delta_\ell \in F$ such that $\delta_0 \neq 0$, $\delta_\ell \neq 0$ and $d_s(r) = \delta_s$ for all r . We assume without loss of generality that $\delta_\ell = 1$ in every constant-coefficient operator. Since every constant-coefficient operator is nonsingular, by our definitions, the kernel $K(D)$ has dimension ℓ ; see discussion preceding Definition 1.3.1.

A particular basis, which we shall call the standard basis, exists for each constant-coefficient operator; see [14, §§13.0-13.1]. The polynomial

$$P(\xi) = \xi^\ell + \delta_{\ell-1} \xi^{\ell-1} + \dots + \delta_0$$

is known as the characteristic polynomial for D . The standard basis is the set of sequences

$$\Lambda = \{(r^m \lambda^r) \in S \mid P(\lambda) = 0 \text{ and } m \in \{0, 1, \dots, \mu(\lambda)-1\}\}$$

where $\mu(\lambda)$ denotes the multiplicity of the root λ . We are using the notation $(r^m \lambda^r)$ here as an alternative way of denoting the sequence

$$(\rho_m, \lambda, 2^m \lambda^2, 3^m \lambda^3, \dots)$$

where $\rho_0 = 1$ and $\rho_m = 0$ when $m > 0$.

The standard basis of a constant-coefficient operator is not necessarily totally ranked; recall Definition 1.3.1. For example, if λ_1 and λ_2 are two distinct zeros of the characteristic polynomial such that $|\lambda_1| = |\lambda_2|$ then $|\lambda_1^r| = |\lambda_2^r|$ for all r . Neither solution dominates the other.

Let us write $a < b$ whenever $a, b \in S$ are such that a is dominated by b . We shall say that a and b are separated (at infinity) if $a < b$ or $b < a$. A totally ranked basis is linearly ordered by $<$ but in general the standard basis is not, as we have just seen. However, we can introduce a linearly ordered partition of the standard basis. We shall proceed to do this.

It is apparent that $(r^m \lambda_1^r) < (r^m \lambda_2^r)$ if, and only if, either (i) $|\lambda_1| < |\lambda_2|$ or (ii) $|\lambda_1| = |\lambda_2|$ and $m_1 < m_2$. Let σ denote the number of roots of $P(\xi) = 0$ of distinct absolute value, and let $\lambda_1, \lambda_2, \dots, \lambda_\sigma$ be representative roots of $P(\xi) = 0$ such that $|\lambda_s| < |\lambda_{s+1}|$ for each s . Define $\mu_s = \max\{\mu(\lambda) \mid |\lambda| = |\lambda_s|\}$. We define the sets

$$\Lambda_s = \{(r^m \lambda^r) \in \Lambda \mid |\lambda| = |\lambda_s|\}, \quad s = 1, 2, \dots, \sigma$$

and, for each s , the sets

$$\Lambda_{s,m} = \{(r^m \lambda^r) \in \Lambda_s \mid \mu(\lambda) \geq m\}, \quad m = 0, 1, \dots, \mu_s - 1.$$

It may be verified readily that the sets $\Lambda_{s,m}$ are disjoint and nonempty and

$$(2.1.1) \quad \Lambda = \bigcup_{s=1}^{\sigma} \bigcup_{m=0}^{\mu_s - 1} \Lambda_{s,m}$$

The sets $\Lambda_{s,m}$ are linearly ordered according to the definition:

$\Lambda_{s,m} < \Lambda_{s',m'}$ if, and only if, either (i) $s < s'$ or (ii) $s = s'$ and $m < m'$.

An illustrative example is afforded by the constant-coefficient equation

$$\begin{aligned}
 y(r+8) - (4+4\sqrt{2})y(r+7) + (15+16\sqrt{2})y(r+6) - (48+12\sqrt{2})y(r+5) \\
 - 12y(r+4) + (192+64\sqrt{2})y(r+3) - (208+256\sqrt{2})y(r+2) + (256+192\sqrt{2})y(r+1) \\
 - 192y(r) = 0 .
 \end{aligned}$$

Its characteristic polynomial, in real factored form, is

$$(2.1.2) \quad (\xi-1)(\xi-2)(\xi+2)(\xi^2-2\sqrt{2}\xi+4)^2(\xi-3) = 0 .$$

The roots of (2.1.2) include the complex numbers $\sqrt{2}(1+i)$ and $\sqrt{2}(1-i)$, each of multiplicity two. The subsets of the linearly ordered partition in this example are

$$\Lambda_{1,0} = \{1^r\}$$

$$\Lambda_{2,0} = \{2^r, (-2)^r, \sqrt{2}(1-i)^r, \sqrt{2}^r(1+i)^r\}$$

$$\Lambda_{2,1} = \{r\sqrt{2}^r(1-i)^r, r\sqrt{2}^r(1+i)^r\}$$

$$\Lambda_{3,0} = \{3^r\} .$$

Note that if $a \in \Lambda_{2,0}$ and $b \in \Lambda_{2,1}$ then $|a(r)/b(r)| = r^{-1}$ when $r \geq 1$.

In contrast, if $a \in \Lambda_s$ and $b \in \Lambda_{s'}$, where $s < s'$ then $|a(r)/b(r)|$ contains a decreasing exponential factor.

In general, if $a, b \in \Lambda$ are such that $a < b$, then the separation ratios $|a(r)/b(r)|$ decrease as the product of a power of r^{-1} and an exponential function in r . If the exponential function is identically equal to one, as will be the case whenever a and b are standard basis solutions which correspond to characteristic roots of equal magnitude, then we shall say that a and b are algebraically separated. Otherwise, we shall say that a and b are exponentially separated.

Figure 5 is obtained by plotting the roots of (2.1.2) in the complex plane and drawing circles through them, centered at the origin. There are three circles, corresponding to the distinct absolute values of the roots.

The double roots are indicated on the circle of radius two; it is helpful to imagine them duplicated on a second circle of radius two that is raised slightly out of the plane. Let us number the four circles, thus arrived at, by increasing three-dimensional distance from the origin.

This determines geometrically the proper linear ordering of the four subsets $\Lambda_{s,m}$ of the partition. The ordering of elements within any one of the subsets is unimportant for our purposes.

Clearly, this geometric indication of the linear ordering of the subsets of the partition can be extended to any constant-coefficient operator.

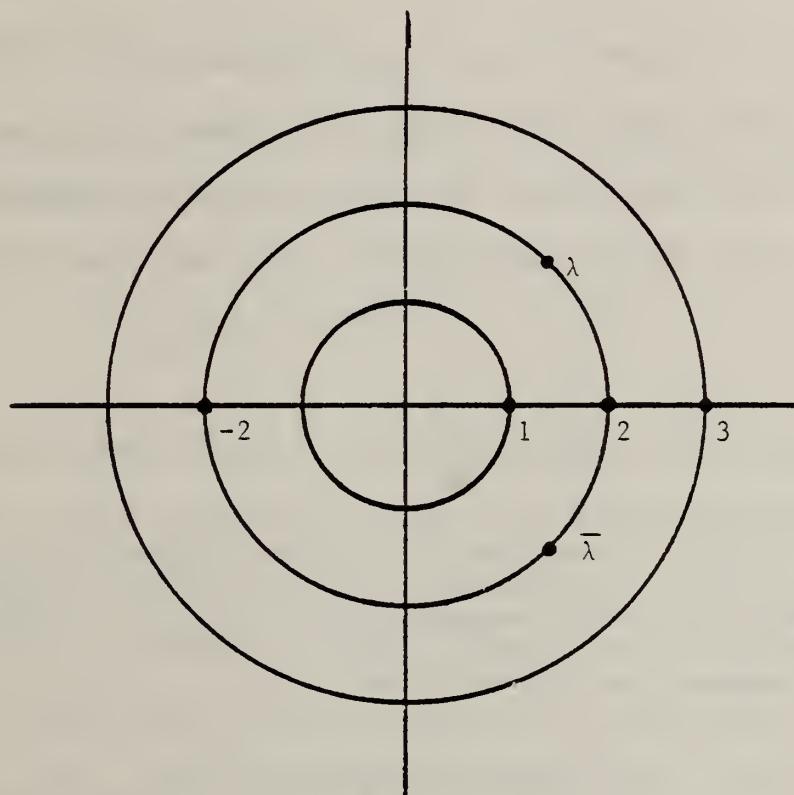


Figure 5. The roots of (2.1.2). The double roots $\sqrt{2}(1\pm i)$ are denoted $\lambda, \bar{\lambda}$.

Theorem 2.1.1 Let (2.1.1) denote the linearly ordered partition of the standard basis for a given constant-coefficient operator. Then:

- i) If $a \in \wedge_{s,m}$ and $b \in \wedge_{s',m'}$, then $a < b$ if, and only if,
 $\wedge_{s,m} < \wedge_{s',m'}$.
- ii) If $a, b \in \wedge_{s,m}$, then $|a(r)| = |b(r)|$ for all r .
- iii) If $a_1, a_2, \dots, a_q, b \in \wedge_{s,m}$ where the a 's are distinct, and $z_1, z_2, \dots, z_q \in \mathbb{F}$ are all nonzero, then

$$(2.1.3) \quad \limsup_{r \rightarrow \infty} \left| \sum_{p=1}^q z_p a_p(r)/b(r) \right| > 0.$$

Proof: Parts (i) and (ii) are trivial consequences of our definitions.

Turning to part (iii), we note that the sequence $\sum_{p=1}^q z_p a_p(r)/b(r)$ is not identically zero (since the a 's are linearly independent). Let $\lambda_1, \lambda_2, \dots, \lambda_q, \lambda$ be the characteristic roots associated with the set $\wedge_{s,m}$ such that

$$a_p(r) = r^m \lambda_p^r, \quad p = 1, 2, \dots, q$$

and $b(r) = r^m \lambda^r$. Since the λ 's lie on the same circle in the complex plane, there exist real constants $\beta_1, \beta_2, \dots, \beta_q, \beta$ all lying in the half-open real interval $[-\pi, \pi]$ such that

$$a_p(r) = r^m |\lambda|^r e^{ir\beta_p}, \quad p = 1, 2, \dots, q$$

and $b(r) = r^m |\lambda|^r \exp(ir\beta)$, $i = \sqrt{-1}$. Thus

$$\sum_{p=1}^q z_p a_p(r)/b(r) = \sum_{p=1}^q z_p e^{ir(\beta_p - \beta)}$$

where $\gamma_p = \beta_p - \beta$. The γ 's are distinct real numbers (modulo 2π)

because the sequences a_1, a_2, \dots, a_q are assumed distinct.

The right member of the last equation represents a so-called almost periodic function of r when r is regarded as a continuous real variable. To complete the proof we appeal to the following lemma, the proof of which is given in the Appendix.[†]

Lemma 2.1.1 Let $\alpha_1, \alpha_2, \dots, \alpha_q$ be nonzero complex numbers and $\gamma_1, \gamma_2, \dots, \gamma_q$ distinct real numbers (modulo 2π). Define the infinite sequence

$$(2.1.4) \quad x(r) = \sum_{p=1}^q \alpha_p e^{ir\gamma_p}, \quad r = 0, 1, 2, \dots .$$

Then $\limsup_{r \rightarrow \infty} |x(r)| > 0$. \square

An equivalent statement to inequality (2.1.3) is that there exists an infinite subsequence of the sequence

$$\sum_{p=1}^q \alpha_p a_p(r)/b(r), \quad r = 0, 1, 2, \dots$$

which is bounded away from zero. Intuitively, this means that it is impossible to create a new sequence with qualitatively different asymptotic behavior merely by forming a linear combination of standard basis solutions of similar asymptotic behavior. Note, however, that unlike the original sequences infinitely many zeros may occur in the new sequence. For example, if 2^r and $(-2)^r$ are sequences in a standard basis, then every other term of the sequence $2^r + (-2)^r$ is zero.

[†]We relegate the proof of Lemma 2.1.1 to the Appendix because the theory of almost periodic functions is not germane to the rest of this thesis.

2.2 Separable Operators

In this section we introduce a subclass of the general set of infinite upper-triangular band matrices \mathcal{D}_ℓ introduced in §1.2, and examine in detail the general structure of the kernel of an operator in this subclass. This will prepare the way for the presentation, in the next section, of a general classification of all sequences in S with respect to a given linear difference operator. In Theorem 2.1.1 we presented three properties of the standard basis of a constant-coefficient operator. These properties suggest the following extension of Definition 1.3.1:

Definition 2.2.1 Let $D \in \mathcal{D}_\ell$ be such that $K(D)$ has dimension ℓ .

Then D will be said to be separable if there exists a basis X for D which can be partitioned into nonempty disjoint subsets $X_1, X_2, \dots, X_\sigma$ in such a way that the following three conditions are all satisfied:

- i) If $x \in X_s$ and $y \in X_t$, where $s < t$, then $\lim_{r \rightarrow \infty} x(r)/y(r) = 0$;
- ii) If $x, y \in X_s$ for some s , then $0 < \liminf_{r \rightarrow \infty} |x(r)/y(r)|$ and $\limsup_{r \rightarrow \infty} |x(r)/y(r)| < \infty$;
- iii) If x is any linear combination of sequences from X_s , other than $x = 0$, and $y \in X_s$, then $\limsup_{r \rightarrow \infty} |x(r)/y(r)| > 0$.

Let D be a separable operator. A basis X for D which is such that

$$(2.2.1) \quad X = \bigcup_{s=1}^{\sigma} X_s,$$

where the X_s satisfy the conditions of Definition 2.2.1 will be said to be optimally ranked. Let

$$(2.2.2) \quad X_s = \{x_{s,1}, x_{s,2}, \dots, x_{s,n_s}\},$$

where the subscript n_s is positive (because X_s is nonempty).

Obviously, we have

$$(2.2.3) \quad \ell = \sum_{s=1}^{\sigma} n_s .$$

In Lemma 2.2.1 below we will show that two distinct optimally ranked bases for a given separable operator have the same number (σ) of disjoint subsets and the same number (n_s) of sequences in corresponding subsets.

An optimally ranked basis exhibits the full range of possible asymptotic behavior of solutions to the corresponding homogeneous difference equation. Conditions (i) and (ii) of Definition 2.2.1 establish the ranking of sequences within the basis. If (2.2.1) is a particular optimally ranked basis, we shall say that $x \in X$ is a sequence of type s provided that $x \in X_s$. More generally, we shall say that $x \in K(D)$ is a sequence of type s provided that $x \neq 0$ and there exist scalars $a_{p,q} \in F$ such that

$$(2.2.4) \quad x = \sum_{p=1}^s \sum_{q=1}^{n_p} a_{p,q} x_{p,q} ,$$

where $a_{s,q} \neq 0$ for some q , $1 \leq q \leq n_s$. Such a representation of x exists and is unique for every nonzero $x \in K(D)$. Again, in consequence of Lemma 2.2.1 below, if we change to a different optimally ranked basis, the type of every sequence in $K(D)$ remains invariant. Condition (iii) of Definition 2.2.1 presages this; it requires that no nontrivial linear combination of basis sequences of the same type may be dominated by a basis sequence of that type.

Let $L(Y)$ denote the linear subspace of S which is spanned by Y where $Y \subset S$ may be any finite subset of sequences. If (2.2.1) is an optimally ranked basis for a separable operator D , then we shall say that the subspace $L(\bigcup_{p=1}^s X_p)$ is the subdominant subspace of type s for D.

These subspaces are independent of the basis; again, see Lemma 2.2.1 below. If j_s denotes the dimension of $L(\bigcup_{p=1}^s X_p)$, then

$$(2.2.5) \quad j_s = \sum_{p=1}^s n_p;$$

compare (2.2.2) and (2.2.3). In addition, we shall say that $\{0\}$, the linear subspace of dimension zero (whose only sequence is the zero sequence) is the subdominant subspace of type 0.

Let U be one of the subdominant subspaces for D . The complementary subspace of U in $K(D)$, i.e., the subspace V of $K(D)$ such that

$$(2.2.6) \quad K(D) = U \oplus V,$$

will be called the corresponding dominant subspace. The symbol \oplus on the right side of (2.2.6) means that $K(D)$ is the direct sum of the subspaces U and V ; that is, every nonzero sequence $x \in K(D)$ is uniquely expressible in one of the forms $x = u$, or $x = v$, or $x = u+v$, where $u \in U$ and $v \in V$. Clearly, if U is the subdominant subspace of type s , $0 \leq s < \sigma$, then $V = L(\bigcup_{p=s+1}^\sigma X_p)$; and if U is the subdominant subspace of type σ , then $V = \{0\}$.

Before stating and proving Lemma 2.2.1, we note that

$$L(\bigcup_{p=1}^s X_p) = L(X_1) \oplus L(X_2) \oplus \cdots \oplus L(X_s)$$

Thus, an alternative characterization of a sequence of type s in $K(D)$ is the following: $x \in K(D)$ is of type s , $1 \leq s \leq \sigma$, provided that

$$x = \sum_{p=1}^s x_p, \quad x_p \in L(X_p), \quad x_s \neq 0;$$

compare (2.2.4).

Lemma 2.2.1 Let

$$X = \bigcup_{s=1}^{\sigma} X_s = \bigcup_{s=1}^{\sigma} \{x_{s,1}, x_{s,2}, \dots, x_{s,m_s}\}$$

and

$$Y = \bigcup_{s=1}^{\tau} Y_s = \bigcup_{s=1}^{\tau} \{y_{s,1}, y_{s,2}, \dots, y_{s,n_s}\}$$

be distinct optimally ranked bases for a separable operator $D \in \mathcal{D}_\ell$.

Then $\sigma = \tau$ and $m_s = n_s$ for all s . Furthermore,

$$L(\bigcup_{p=1}^s X_p) = L(\bigcup_{p=1}^s Y_p), \quad s = 1, 2, \dots, \sigma,$$

and the type of every nonzero sequence $z \in K(D)$ is the same with respect to X as it is with respect to Y .

Proof Let $u \in X_s$ and $v \in X_t$, so that relative to X the types of u and v are s and t . Let the types of u and v relative to Y be s' and t' . Then there exist unique representations

$$u = \sum_{j=1}^{s'} u_j, \quad u_j \in L(Y_j), \quad u_s \neq 0$$

and

$$v = \sum_{j=1}^{t'} v_j, \quad v_j \in L(Y_j), \quad v_{t'} \neq 0.$$

Furthermore, by condition (i) of Definition 2.2.1, we have, as $r \rightarrow \infty$,

$$(2.2.7) \quad u(r) = u_s, (r) + o\{y(r)\}, \quad y \in Y_s,$$

and

$$(2.2.8) \quad v(r) = v_{t'}(r) + o\{y(r)\}, \quad y \in Y_{t'}, .$$

i) First we prove that $s' = t'$ if $s = t$, i.e., if u and v are of the same type with respect to X , then they are also of the same type with respect to Y .

Since $s = t$, the sequence $u(r)/v(r)$ is asymptotically bounded away from zero; compare part (ii) of Definition 2.2.1. Suppose that $s' < t'$. We shall obtain a contradiction by showing that, under this assumption, $u(r)/v(r)$ has an infinite subsequence which converges to zero. Choose any $y \in Y_{t'}, .$ Then we have

$$\lim_{r \rightarrow \infty} \frac{u(r)}{y(r)} = 0 .$$

From (2.2.8), we have

$$\frac{v(r)}{y(r)} = \frac{v_{t'}(r)}{y(r)} + o(1)$$

and, using the triangle inequality, an elementary property of the limit superior, and condition (iii) of Definition 2.2.1, we obtain

$$\limsup_{r \rightarrow \infty} \left| \frac{v(r)}{y(r)} \right| = \limsup_{r \rightarrow \infty} \left| \frac{v_{t'}(r)}{y(r)} \right| > 0 .$$

Thus there exists an infinite subsequence of $v(r)/y(r)$ that is bounded away from zero, and therefore also a subsequence of

$$\frac{u(r)}{v(r)} = \frac{u(r)/y(r)}{v(r)/y(r)}$$

that converges to zero. This supplies the needed contradiction.

A similar contradiction is obtained if we suppose that $s' > t'$. Hence $s' = t'$.

ii) Next we prove that $s' < t'$ if $s < t$.

Since $s < t$, the sequence $u(r)/v(r)$ converges to zero. Suppose that $s' \geq t'$. We shall derive a contradiction by showing that, under this assumption, $u(r)/v(r)$ has a subsequence which does not converge to zero. Choose any $y \in Y_{s'}$. First, by using (2.2.7) and an argument similar to that used in part (i) of this proof, we have

$$\limsup_{r \rightarrow \infty} \left| \frac{u(r)}{y(r)} \right| > 0.$$

Thus there exists an infinite subsequence of $u(r)/y(r)$ that is bounded away from zero.

Assume now that $s' > t'$. Then

$$\lim_{r \rightarrow \infty} \frac{v(r)}{y(r)} = 0.$$

In this case it follows that there exists a subsequence of

$$\frac{u(r)}{v(r)} = \frac{u(r)/y(r)}{v(r)/y(r)}$$

that diverges to ∞ , which supplies the needed contradiction.

Alternatively, assume that $s' = t'$. Using (2.2.8), we have for sufficiently large r

$$\left| \frac{v(r)}{y(r)} \right| \leq \left| \frac{v_{t'}(r)}{y(r)} \right| + 1.$$

Furthermore, using condition (ii) of Definition 2.2.1, we see that there exist $r_0 \geq 0$ and $\beta > 0$ such that

$$\left| \frac{v_{t'}(r)}{y(r)} \right| \leq \beta, \quad r \geq r_0.$$

Therefore, for sufficiently large r , we have

$$\left| \frac{v(r)}{y(r)} \right| \leq \beta + 1$$

and

$$\left| \frac{u(r)}{v(r)} \right| \geq (1+\beta)^{-1} \left| \frac{u(r)}{y(r)} \right| .$$

It follows that $u(r)/v(r)$ has a subsequence that is bounded away from zero, which again supplies the needed contradiction.

iii) Next, we prove that $\sigma = \tau$ and $m_s = n_s$ for all s , and also that $L(\bigcup_{p=1}^s X_p) = L(\bigcup_{p=1}^s Y_p)$ for all s .

Part (i) of this proof implies that for each s , $1 \leq s \leq \sigma$, the type relative to Y of every sequence in X_s is the same, say t_s . Using part (ii), we have

$$(2.2.9) \quad 1 \leq t_1 < t_2 < \cdots < t_\sigma \leq \tau .$$

Furthermore, that $s \leq t_s$ for all s , $1 \leq s \leq \sigma$, is easily proved by induction. In particular, $\sigma \leq t_\sigma$ and, using (2.2.9), we see that $\sigma \leq \tau$. Inverting the roles of X and Y and repeating the argument, we see that $\tau \leq \sigma$. Thus $\sigma = \tau$, and (2.2.9) implies $t_s = s$ for all s .

For each s , it follows that

$$L(X_1 \cup \cdots \cup X_s) \subset L(Y_1 \cup \cdots \cup Y_s) ,$$

and by symmetry,

$$L(Y_1 \cup \dots \cup Y_s) \subset L(X_1 \cup \dots \cup X_s) .$$

Thus

$$L(X_1 \cup \dots \cup X_s) = L(Y_1 \cup \dots \cup Y_s)$$

for each s . In particular, $L(X_1) = L(Y_1)$. Since X_1 and Y_1 are bases of the same linear space, we have $m_1 = n_1$. An easy induction completes the proof of this part.

iv) To complete the proof, let $z \in K(D)$ be a nonzero sequence such that

$$z = \sum_{p=1}^s u_p , \quad u_s \neq 0 , \quad u_p \in L(X_p) ,$$

and

$$z = \sum_{p=1}^t v_p , \quad v_t \neq 0 , \quad v_p \in L(Y_p) .$$

Assume $s < t$, and let $y \in Y_t$. Then

$$\lim_{r \rightarrow \infty} \frac{v_t(r)}{y(r)} = \lim_{r \rightarrow \infty} \left[\sum_{p=1}^s \frac{u_p(r) - v_p(r)}{y(r)} - \sum_{p=s+1}^{t-1} \frac{v_p(r)}{y(r)} \right] = 0 .$$

But this result contradicts condition (iii) of Definition 2.2.1. Hence we must have $s \geq t$. A similar contradiction is obtained if we assume $s > t$. Therefore $s = t$ and the proof of Lemma 2.2.1 is complete. \square

In general, the actual determination of optimally ranked bases for variable coefficient difference operators is a difficult problem. Some results have been given by Wimp [26]. This reference extends earlier work by Birkhoff and Trjitzinsky [2, 3] on the analytic theory of singular difference equations whose coefficients possess asymptotic expansions

of a prescribed form. All equations with coefficients rational in r are included, for example. Wimp's analysis provides a means of constructing a basis, which he calls a canonical set; see [26, Theorem 3.3 and Definition 3.6]. The canonical set is analogous to our optimally ranked basis. More specifically, there exists a basis $\{y_1, y_2, \dots, y_\ell\}$ such that, for each $s = 1, 2, \dots, \ell$,

$$y_s(r) = c_s M_s(r)\{1+o(1)\} \quad \text{as } r \rightarrow \infty$$

where $c_s \neq 0$ and

$$M_s(r) = e^{Q_s(r)} r^{\theta_s} (\ln r)^{p_s}.$$

Here θ_s is a complex number and p_s is a positive integer. Q_s is of the form

$$Q_s(r) = \mu_0 r \ln r + \mu_1 r + \mu_2 r^{1-1/\rho} + \dots + \mu_\rho r^{1/\rho}$$

where ρ is an integer, $\rho \geq 1$, and $\mu_0, \mu_1, \dots, \mu_\rho$ are complex numbers.

The rich variety of possible separation ratios of solutions as $r \rightarrow \infty$, compared with the algebraic or exponential separation of solutions in the constant-coefficient case, described in §2.1 above, is evident.

2.3 The General Classification

In §2.2 we showed that every nonzero sequence in the kernel of a separable operator has a unique type with respect to that particular operator. In this section we extend this classification to every nonzero sequence in S . The following definition should be compared with Definition 1.3.2.

Definition 2.3.1 Let $D \in \mathcal{D}_\ell$ be separable with optimally ranked basis $X = \bigcup_{s=1}^{\sigma} X_s$. Then, for each $y \in S$, we say that $\text{type}(y) = s$, or more fully, the type of y with respect to D is s , provided that $0 < s < \sigma$ and the following two conditions are satisfied:

$$\text{i)} \quad \text{If } x \in X_{s+1} \text{ then } \lim_{r \rightarrow \infty} \frac{y(r)}{x(r)} = 0 ;$$

$$\text{ii)} \quad \text{If } x \in X_s \text{ then } \infty \geq \limsup_{r \rightarrow \infty} \left| \frac{y(r)}{x(r)} \right| > 0 .$$

If (i) is satisfied for $s = 0$, then (ii) does not apply, and we define $\text{type}(y) = 0$. Similarly, if (ii) is satisfied for $s = \sigma$, (i) does not apply, and we define $\text{type}(y) = \sigma$.

The classification of sequences introduced in Definition 2.3.1 allows us to treat both homogeneous and inhomogeneous difference equations in a uniform manner when the difference operator is separable. The kernel of the operator contains every solution of the homogeneous equation, by definition. But solutions of the inhomogeneous equation with arbitrary right-hand side may lie anywhere in S . This is why a classification of all sequences in S is desirable.

Theorem 2.3.1 Every sequence in S possesses an unambiguous type with respect to each separable operator in \mathcal{D}_ℓ .

Proof (i) First we show that the type with respect to a fixed optimally

ranked basis is determined uniquely by Definition 2.3.1 for every nonzero sequence in S .

Let $z \in S$. Assuming that

$$(2.3.1) \quad \limsup_{r \rightarrow \infty} \left| \frac{z(r)}{x(r)} \right| > 0$$

for at least one sequence $x \in X$, let $\hat{x} \in X$ be a sequence of largest type such that (2.3.1) is satisfied. Let $s = \text{type } (\hat{x})$. Then (2.3.1) is satisfied for every $x \in X_s$, since

$$\left| \frac{z(r)}{x(r)} \right| = \left| \frac{z(r)}{\hat{x}(r)} \right| \left| \frac{\hat{x}(r)}{x(r)} \right|$$

and $\liminf_{r \rightarrow \infty} |\hat{x}(r)/x(r)| > 0$ as a consequence of condition (ii) of Definition 2.2.1.

If $s = \sigma$, then $\text{type}(z) = \sigma$ according to Definition 2.3.1. Therefore suppose that $s < \sigma$. By the maximality of \hat{x} we see that $\limsup_{r \rightarrow \infty} |z(r)/x(r)| = 0$ whenever $\text{type } (x) > s$ or, equivalently,

$$(2.3.2) \quad \lim_{r \rightarrow \infty} \frac{z(r)}{x(r)} = 0 \text{ whenever } \text{type } (x) > s.$$

Accordingly, $\text{type } (z) = s$.

To conclude the proof of part (i), let us suppose that (2.3.1) is not satisfied for any $x \in X$. Then (2.3.2) is satisfied for every $x \in X$, and in particular for every $x \in X_1$. Then $\text{type } (z) = 0$.

(ii) In this part we prove that if $X = \bigcup_{s=1}^{\sigma} X_s$ and $X' = \bigcup_{s=1}^{\sigma} X'_s$ are two distinct optimally ranked bases for D then the type of every sequence in S is the same with respect to X and X' .

Choose any s , $1 \leq s \leq \sigma$, and choose $x \in X_s$, $x' \in X'_s$. There

exist $x_p \in X_p$, $1 \leq p \leq s$, such that $x' = x_1 + \dots + x_s$; see Lemma 2.2.1.

Furthermore, we have

$$\frac{x'(r)}{x(r)} = o(1) + \frac{x_s(r)}{x(r)} \quad \text{as } r \rightarrow \infty.$$

Since x_s is a linear combination of sequences of type s , a brief computation and application of condition (ii) of Definition 2.2.1 yields

$$\limsup_{r \rightarrow \infty} \left| \frac{x_s(r)}{x(r)} \right| < \infty.$$

It follows that

$$(2.3.3) \quad \limsup_{r \rightarrow \infty} \left| \frac{x'(r)}{x(r)} \right| < \infty$$

$$\text{and } \liminf_{r \rightarrow \infty} \left| \frac{x(r)}{x'(r)} \right| > 0 \text{ whenever type}(x') = \text{type}(x).$$

Since the roles of x and x' are interchangeable, we also have

$$(2.3.4) \quad \limsup_{r \rightarrow \infty} \left| \frac{x(r)}{x'(r)} \right| < \infty$$

$$\text{and } \liminf_{r \rightarrow \infty} \left| \frac{x'(r)}{x(r)} \right| > 0 \text{ whenever type}(x') = \text{type}(x).$$

Let $z \in S$ and suppose $\text{type}(z) = 0$ with respect to X . Thus $z(r)/x(r) \rightarrow 0$ for every $x \in X$. Choose $x' \in X'$. Let $s = \text{type}(x')$ and choose $x \in X_s$. Then

$$\left| \frac{z(r)}{x'(r)} \right| = \left| \frac{z(r)}{x(r)} \right| \left| \frac{x(r)}{x'(r)} \right| \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

by virtue of (2.3.4). This proves $\text{type}(z) = 0$ with respect to X' .

Let $z \in S$ and suppose $\text{type}(z) = \sigma$ with respect to X . Thus $\limsup_{r \rightarrow \infty} |z(r)/x(r)| > 0$ for every $x \in X$. Choose $x' \in X'$. Let $s = \text{type}(x')$ and choose $x \in X_s$. Then the sequence

$$\left| \frac{z(r)}{x'(r)} \right| = \left| \frac{z(r)}{x(r)} \right| \left| \frac{x(r)}{x'(r)} \right|$$

satisfies $\limsup_{r \rightarrow \infty} |z(r)/x'(r)| > 0$ by virtue of (2.3.3). This proves $\text{type}(z) = \sigma$ with respect to X' .

Finally, let $z \in S$ and suppose $\text{type}(z) = s$, $0 < s < \sigma$. Thus $z(r)/x(r) \rightarrow 0$ for every $x \in X_{s+1} \cup \dots \cup X_\sigma$ and $\limsup_{r \rightarrow \infty} |z(r)/x(r)| > 0$ for every $x \in X_s$. It can be verified that $z(r)/x'(r) \rightarrow 0$ for every $x' \in X_{s+1} \cup \dots \cup X_\sigma$ and that $\limsup_{r \rightarrow \infty} |z(r)/x'(r)| > 0$ for every $x' \in X'_s$ using arguments similar to the ones used in the preceding two paragraphs. \square

CHAPTER 3. THE GENERAL ALGORITHM

3.0 Preliminaries and Overview

In this chapter we fix our attention on a particular linear difference equation

$$(3.0.1) \quad Dy(r) = g(r), \quad r = 0, 1, 2, \dots$$

assuming that $D \in \mathcal{D}_\ell$ is separable and $g \in S$. More specifically, we are interested in computing an approximation y_n of a particular solution y of (3.0.1) which is valid over some finite subsequence $y^{i,m}$. i and m given.

Let σ be the number of distinct types in any optimally ranked basis for D . We assume that y is known to be a sequence of type t with respect to D , where $0 \leq t \leq \sigma$. We also assume that initial values $y(i), y(i+1), \dots, y(i+j-1)$ are known, where j is the dimension of the subdominant subspace of type t for D . The extension of Theorem 1.3.1, from totally separable to separable operators, is given in §3.1. Accordingly, these initial values (in the presence of one additional condition) suffice to determine the solution uniquely.

Let $k = \ell - j$. The approximating sequence y_n is defined as the solution of (3.0.1) which satisfies the conditions

$$(3.0.2) \quad y_n(i+r) = y(i+r), \quad r = 0, 1, \dots, j-1$$

and

$$(3.0.3) \quad y_n(n+r) = 0, \quad r = 0, 1, \dots, k-1.$$

Since $y^{i,m}$ is to be approximated by $y_n^{i,m}$, it is clear that the value

of n must exceed m . The existence and uniqueness of the approximating sequences y_n , for all sufficiently large n , and the convergence of the sequence of approximate values $y_n(r)$ to $y(r)$ for each value of r in the range $i + j \leq r < \infty$, are proved under appropriate conditions in §3.2.

The general algorithm, which is valid for any separable operator, reduces to the algorithm presented in §1.3 for totally separable operators. We again assume that the separable operator is (i,j) - factorizable, just as we did for totally separable operators. The linear system (1.3.5) is solved by the factorization method by means of a forward elimination stage followed by a back substitution stage; see §1.3. The expansions (1.3.19) and (1.3.20), which are used to estimate the optimal value of n , are unchanged for separable operators. In §3.3 we discuss the stability of the forward elimination stage, which was omitted in §1.3. We also extend Theorem 1.3.2 to separable operators, showing that the back substitution is stable. Finally, we note that our remarks on the stability of the forward recurrence solution of the adjoint equation, whose solution enters in the truncation error expansion of §1.3, apply to separable operators as well as totally separable operators.

We conclude this introductory section by fixing some notation for later use in this chapter. Let

$$(3.0.4) \quad X = \bigcup_{s=1}^{\sigma} X_s$$

be an optimally ranked basis for D , where

$$(3.0.5) \quad X_s = \{x_{s,1}, x_{s,2}, \dots, x_{s,n_s}\}, \quad s = 1, 2, \dots, \sigma.$$

Then the dimension of the subdominant subspace of type t for D is

$$(3.0.6) \quad j = \sum_{s=1}^t n_s .$$

Define the set

$$(3.0.7) \quad U = \{u_1, u_2, \dots, u_j\} = \bigcup_{s=1}^t X_s$$

by means of the correspondence

$$x_{p,q} = u_{n_1 + n_2 + \dots + n_{p-1} + q} .$$

When $t = 0$ it is understood that $j = 0$ and U is empty. Similarly, define the set

$$(3.0.8) \quad V = \{v_1, v_2, \dots, v_k\} = \bigcup_{s=t+1}^{\sigma} X_s ;$$

when $t = \sigma$ it is understood that V is empty.

3.1 Existence and Uniqueness

Let y be a solution of (3.0.1) of type t such that

$$(3.1.1) \quad y(i+r) = \gamma_r, \quad r = 0, 1, \dots, j-1$$

for some $i \geq 0$, where j is given by (3.0.6). Using the notation $|U|(i)$ for the Casoratian of U at i , we also suppose that

$$(3.1.2) \quad |U|(i) \neq 0;$$

see (3.0.7). Note that for every i (3.1.2) is satisfied vacuously if $t = 0$, and also (by virtue of Cásorati's theorem [14,§12.11]) if $t = \sigma$ and D is nonsingular. The following theorem generalizes Theorem 1.3.1.

Theorem 3.1.1 If (3.1.2) is valid then there exists a unique solution of (3.0.1) of type t such that (3.1.1) is satisfied.

Proof: Suppose $t = 0$. Then $j = 0$ and we must prove that without any specified initial values y is uniquely determined. Let $v \in S$ be a solution of (3.0.1), other than y , such that $\text{type}(v) = 0$. Since $y - v \in K(D)$, there exist scalars $\alpha_{p,q} \in F$ such that

$$y - v = \sum_{p=1}^{\sigma} \sum_{q=1}^{n_p} \alpha_{p,q} x_{p,q};$$

see (3.0.5). At least one of the $\alpha_{p,q}$ is nonzero; let the largest value of p such that $\alpha_{p,q} \neq 0$ for some q , $1 \leq q \leq n_p$, be $p = s$.

Then

$$\lim_{r \rightarrow \infty} \frac{y(r) - v(r)}{x_{s,1}(r)} = 0$$

because $\text{type}(y) = \text{type}(v) = 0$. By part (i) of Definition 2.2.1, we

have

$$\lim_{r \rightarrow \infty} \sum_{p=1}^{s-1} \sum_{q=1}^{n_p} \frac{\alpha_{p,q} x_{p,q}(r)}{x_{s,1}(r)} = 0 .$$

It follows from the above three equations that

$$\lim_{r \rightarrow \infty} \sum_{q=1}^{n_s} \frac{\alpha_{s,q} x_{s,q}(r)}{x_{s,1}(r)} = 0 .$$

But this result contradicts part (iii) of Definition 2.2.1; therefore we conclude that $y = v$.

Alternatively suppose $t > 0$. Let $v \in S$ be a sequence of type t which satisfies (3.0.1) and is coincident with $y(i+r)$ for $r = 0, 1, \dots, j-1$. Then there exist scalars $\alpha_{p,q} \in F$ such that

$$v = y + \sum_{p=1}^{\sigma} \sum_{q=1}^{n_p} \alpha_{p,q} x_{p,q} .$$

Let the largest value of p in this sum such that at least one of the scalars $\alpha_{p,q}$ is nonzero be $p = s$. Then $s \leq t$, because $s > t$ would imply

$$\lim_{r \rightarrow \infty} \frac{v(r) - y(r)}{x_{s,1}(r)} = 0$$

(since type $(v) = \text{type } (y) = t$) and

$$\lim_{r \rightarrow \infty} \sum_{p=1}^{s-1} \sum_{q=1}^{n_p} \frac{\alpha_{p,q} x_{p,q}(r)}{x_{s,1}(r)} = 0$$

(by part (i) of Definition 2.2.1). This gives

$$\lim_{r \rightarrow \infty} \sum_{q=1}^n \frac{a_{s,q} x_{s,q}(r)}{x_{s,1}(r)} = 0 ,$$

which is a contradiction.

Thus, $a_{p,q} = 0$ whenever $p > s$. The $a_{p,q}$ having $p \leq s$ satisfy the system of linear equations

$$\sum_{p=1}^s \sum_{q=1}^{n_p} a_{p,q} x_{p,q}(i+r) = 0 , \quad r = 0, 1, \dots, j-1 .$$

This system is nonsingular by the condition $|U|(i) \neq 0$, so we have

$a_{p,q} = 0$ for all p and q . Thus $v(r) = y(r)$ for all r and the theorem is proved. \square

Theorem 3.1.2 If D is nonsingular then, for each $i \geq 0$, $|U|(i+r) \neq 0$ for at least one value of r in the range $0 \leq r \leq \ell - j$.

Proof: Since D is nonsingular, Casorati's theorem is valid, i.e., $|X|(i) \neq 0$ for all i . Theorem 3.1.2 is then an immediate consequence of Laplace's general theorem on the expansion of determinants [15, §93]. \square

Theorem 3.1.2 implies there is no shortage of points i which satisfy condition (3.1.2) when the difference operator is nonsingular. Our next goal is to show that any optimally ranked basis may be used to locate an admissible value of i .

Theorem 3.1.3 If $|U|(i) \neq 0$ for some fixed $i \geq 0$, then the leading principal minor of order j of the Casoratian evaluated at i of every optimally ranked basis for D is nonzero.

Proof: If $j = 0$ then there is nothing to prove, so let us assume $j > 0$. Also, let us assume temporarily that $j < \ell$. Let X' be an optimally ranked basis distinct from X , where

$$X' = \bigcup_{s=1}^{\sigma} X'_s$$

and

$$X'_s = \{x'_{s,1}, x'_{s,2}, \dots, x'_{s,n_s}\}, \quad s = 1, 2, \dots, \sigma;$$

compare (3.0.4), (3.0.5). Define the sets

$$U' = \{u'_1, u'_2, \dots, u'_j\} = \bigcup_{s=1}^t X'_s$$

and

$$V' = \{v'_1, v'_2, \dots, v'_k\} = \bigcup_{s=t+1}^{\sigma} X'_s.$$

There exist scalars $\alpha_{p,q} \in F$ such that

$$u'_p = \sum_{q=1}^j \alpha_{p,q} u_q + \sum_{q=j+1}^{\ell} \alpha_{p,q} v_{q-j}, \quad p = 1, 2, \dots, j,$$

and

$$v'_{p-j} = \sum_{q=1}^j \alpha_{p,q} u_q + \sum_{q=j+1}^{\ell} \alpha_{p,q} v_{q-j}, \quad p = j+1, j+2, \dots, \ell.$$

The matrix of coefficients $(\alpha_{p,q})$ is nonsingular because both X and X' are bases of the finite-dimensional linear space $K(D)$. Furthermore, from the definition of an optimally ranked basis it is easy to show that $(\alpha_{p,q})$ is block lower triangular with the s -th diagonal block having order n_s . Because of the block triangular structure, the determinant of $(\alpha_{p,q})$ is equal to the product of the determinants of the diagonal blocks. And because the determinant of $(\alpha_{p,q})$ is nonzero, so is the determinant of every diagonal block. Therefore, the leading principal minor of order j of the determinant of $(\alpha_{p,q})$ is nonzero. Let Δ_j denote this minor.

Next, consider the Casorati matrix $[X](i)$. It can be written in block form as

$$[X](i) = \begin{bmatrix} [U](i) & * \\ \dots & \dots \\ * & [V](i+j) \end{bmatrix}$$

where the actual form of the blocks indicated by asterisks is unimportant. Similarly, we write

$$[X'](i) = \begin{bmatrix} [U'](i) & * \\ \dots & \dots \\ * & [V'](i+j) \end{bmatrix}.$$

If

$$(\alpha_{p,q}) = \begin{bmatrix} A_{11} & 0 \\ \dots & \dots \\ A_{21} & A_{22} \end{bmatrix}$$

is the corresponding block form of $(\alpha_{p,q})$, we verify readily that

$$[X'](i) = \begin{bmatrix} [U](i) & * \\ \dots & \dots \\ * & [V](i+j) \end{bmatrix} \begin{bmatrix} A_{11}^{\text{Tr}} & A_{21}^{\text{Tr}} \\ \dots & \dots \\ 0 & A_{22}^{\text{Tr}} \end{bmatrix}.$$

The leading principal minor of order j of $|X'|(i)$ is evidently given by

$$|U'|(i) = |U|(i) \cdot \det(A_{11}).$$

Since $|U|(i) \neq 0$, by hypothesis, and $\det(A_{11}) = \Delta_j \neq 0$ we have $|U'|(i) \neq 0$. This completes the proof for $j < \ell$.

Finally, we note that the proof in the case $j = \ell$ follows by an argument analogous to that just used for $j < \ell$, except that there is no partitioning of matrices since V and V' are empty. \square

3.2 Convergence

Assume[†]

$$(3.2.1) \quad |U|(i) \neq 0$$

and let y be a solution of type t of (3.0.1), where $0 \leq t \leq \sigma$. For each $n \geq m + 1$ such that a solution y_n of (3.0.1) exists satisfying both (3.0.2) and (3.0.3), there exist scalars $\alpha_1(n), \alpha_2(n), \dots, \alpha_j(n)$ and $\beta_1(n), \beta_2(n), \dots, \beta_k(n)$ such that

$$(3.2.2) \quad y_n = y + \sum_{s=1}^j \alpha_s(n) u_s + \sum_{s=1}^k \beta_s(n) v_s.$$

In the case $t = \sigma$, we have

$$y_n = y + \sum_{s=1}^{\ell} \alpha_s(n) u_s.$$

Then we prove readily, using (3.0.2) and (3.2.1), that $\alpha_s(n) = 0$ for $s = 1, 2, \dots, \ell$. Thus for each n , $y_n(r) = y(r)$ for $r = i, i+1, \dots$.

Therefore we restrict the ensuing discussion to the cases in which $0 \leq t < \sigma$.

Consider the set C of all subsets of k distinct sequences from X . We have $k > 0$ by our assumption that $t < \sigma$. Obviously, $V = \{v_1, v_2, \dots, v_k\} \in C$. We shall say that the optimally ranked basis X is j-normal provided that $|V|(r) \neq 0$ for all sufficiently large r , and

$$(3.2.3) \quad |\tilde{V}|(r) = o\{|V|(r)\} \text{ as } r \rightarrow \infty$$

for every $\tilde{V} \in C - \{V\}$. When $t = 0$, $C = \{V\}$ and X is automatically

[†]The reader is referred to §3.0 for notation and underlying assumptions.

j -normal (0 -normal). When $t > 0$ the sets \tilde{V} contain at least one sequence from $U = \{u_1, u_2, \dots, u_j\}$. In this case (3.2.3) expresses the quite reasonable condition that whenever one or more of the "dominant" solutions v_1, v_2, \dots, v_k is replaced in the Casoratian $|V|$ by a "sub-dominant" solution, then the resulting Casoratian is dominated by $|V|(r)$ as $r \rightarrow \infty$.

We now state and prove the convergence theorem. A similar result, but only for homogeneous linear difference equations, is given by Zahar [27, Th. 5.1].

Theorem 3.2.1 Assume X is a j -normal optimally ranked basis such that (3.2.1) is satisfied and $y(r)$ is a solution of (3.0.1) of type t . Also assume

$$(3.2.4) \quad \lim_{r \rightarrow \infty} \frac{|V_s(y)|(r)}{|V|(r)} = 0, \quad s = 1, 2, \dots, k$$

where $V_s(y) = \{v_1, \dots, v_{s-1}, y, v_{s+1}, \dots, v_k\}$. Then for sufficiently large n there exists a unique solution y_n of (3.0.1) which satisfies (3.0.2) and (3.0.3). Furthermore,

$$(3.2.5) \quad \lim_{n \rightarrow \infty} y_n(r) = y(r)$$

for each fixed value of $r \geq i + j$.

Proof: From (3.2.2), (3.0.2) and (3.0.3) we derive the linear system

$$\begin{bmatrix} u_1(i) & \dots & u_j(i) & : & v_1(i) & \dots & v_k(i) \\ \vdots & & \vdots & : & \vdots & & \vdots \\ u_1(i+j-1) & \dots & u_j(i+j-1) & : & v_1(i+j-1) & \dots & v_k(i+j-1) \\ \vdots & \dots & \vdots & : & \vdots & \dots & \vdots \\ u_1(n) & \dots & u_j(n) & : & v_1(n) & \dots & v_k(n) \\ \vdots & & \vdots & : & \vdots & & \vdots \\ u_1(n+k-1) & \dots & u_j(n+k-1) & : & v_1(n+k-1) & \dots & v_k(n+k-1) \end{bmatrix} \begin{bmatrix} \alpha_1(n) \\ \vdots \\ \alpha_j(n) \\ \beta_1(n) \\ \vdots \\ \beta_k(n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dots \\ -y(n) \\ \vdots \\ -y(n+k-1) \end{bmatrix}.$$

Although the block structure indicated in this system disappears when $t = 0$, this modification[†] causes no difficulty in the following proof.

Suppose V were such that the upper right block of the matrix is zero (again, this is satisfied vacuously if $t = 0$). Then the linear system becomes

$$\begin{bmatrix} u_1(i) & \dots & u_j(i) & : & 0 & \dots & 0 \\ \vdots & & \vdots & : & \vdots & & \vdots \\ u_1(i+j-1) & \dots & u_j(i+j-1) & : & 0 & \dots & 0 \\ \vdots & \dots & \vdots & : & \vdots & \dots & \vdots \\ u_1(n) & \dots & u_j(n) & : & v_1(n) & \dots & v_k(n) \\ \vdots & & \vdots & : & \vdots & & \vdots \\ u_1(n+k-1) & \dots & u_j(n+k-1) & : & v_1(n+k-1) & \dots & v_k(n+k-1) \end{bmatrix} \begin{bmatrix} \alpha_1(n) \\ \vdots \\ \alpha_j(n) \\ \vdots \\ \beta_1(n) \\ \vdots \\ \beta_k(n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ -y(n) \\ \vdots \\ -y(n+k-1) \end{bmatrix} .$$

The determinant of this system is $|U|(i) \cdot |V|(n)$, and it is nonzero for all sufficiently large n , in view of our assumptions. Using Cramer's rule and Laplace's general theorem on the expansion of determinants [15, §93], we then find that

$$\alpha_1(n) = \alpha_2(n) = \dots = \alpha_j(n) = 0$$

and

$$\beta_s(n) = -\frac{|v_s(y)|}{|V|(n)}, \quad s = 1, 2, \dots, k .$$

Therefore, for all r and for all sufficiently large n , we have

$$(3.2.6) \quad y_n(r) = y(r) - \sum_{s=1}^k \frac{|v_s(y)|}{|V|(n)} v_s(r) .$$

[†]The other case in which the block structure disappears, which is $t = \sigma$, was disposed of earlier.

Then (3.2.5) follows from (3.2.4).

In order to complete the proof, let us introduce the sequences

$$(3.2.7) \quad v'_p = v_p + \gamma_{p,1} u_1 + \cdots + \gamma_{p,j} u_j, \quad p = 1, 2, \dots, k,$$

where the $\gamma_{p,q}$ are to be chosen in such a way that

$$v'_p(i+r) = 0, \quad r = 0, 1, \dots, j-1.$$

Thus the $\gamma_{p,q}$ have to satisfy the linear systems

$$\begin{bmatrix} u_1(i) & \dots & u_j(i) \\ \vdots & \ddots & \vdots \\ u_1(i+j-1) & \dots & u_j(i+j-1) \end{bmatrix} \begin{bmatrix} \gamma_{p,1} \\ \vdots \\ \gamma_{p,j} \end{bmatrix} = \begin{bmatrix} -v_p(i) \\ \vdots \\ -v_p(i+j-1) \end{bmatrix}, \quad p = 1, 2, \dots, k,$$

each of which is nonsingular because $|U|(i) \neq 0$, by hypothesis. Let $V' = \{v'_1, v'_2, \dots, v'_k\}$. Clearly it suffices to show that $U \cup V'$ is a j -normal optimally ranked basis for D such that

$$(3.2.8) \quad \lim_{r \rightarrow \infty} \frac{|V'(y)|_s(r)}{|V'(r)} = 0, \quad s = 1, 2, \dots, k.$$

Since v_p dominates every sequence in U , we have from (3.2.7)

$$\frac{v'_p(r)}{v_p(r)} \rightarrow 1 \quad \text{as } r \rightarrow \infty$$

for each p , $1 \leq p \leq k$. Therefore v'_p has the same type as v_p .

Furthermore, we may verify the three conditions of Definition 2.2.1, as follows:

$$i) \quad \frac{u(r)}{v'_p(r)} = \frac{u(r)}{v_p(r)} \cdot \frac{v_p(r)}{v'_p(r)} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

for every $u \in U$, and

$$\frac{v'_p(r)}{v'_q(r)} = \frac{v'(r)}{v_p(r)} \cdot \frac{v_p(r)}{v_q(r)} \cdot \frac{v_q(r)}{v'(r)} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

whenever type $(v'_p) < \text{type } (v'_q)$;

$$\text{ii) } \frac{v'_p(r)}{v'_q(r)} = \frac{v'(r)}{v_p(r)} \cdot \frac{v_p(r)}{v_q(r)} \cdot \frac{v_q(r)}{v'(r)} \sim \frac{v_p(r)}{v_q(r)} \quad \text{as } r \rightarrow \infty$$

whenever type $(v'_p) = \text{type } (v'_q)$;

(iii) Let $v'_p, v'_{p+1}, \dots, v'_q \in V'$ be all the sequences of one type,

and let v' be any fixed one of these. Let the corresponding unprimed sequences be $v, v_p, v_{p+1}, \dots, v_q \in V$. Take $\delta_p, \delta_{p+1}, \dots, \delta_q \in F$, not all zero. For each $s = p, p+1, \dots, q$ we have

$$\frac{\delta_s v'_s}{v'} = \frac{\delta_s(v_s + \gamma_{s,1}u_1 + \dots + \gamma_{s,j}u_j)}{v'} = \frac{\delta_s v_s}{v'} + o(1)$$

(since u_1, u_2, \dots, u_j are of lower type than $v' = v\{1+o(1)\}$). Hence

$$\frac{\delta_p v'_p + \dots + \delta_q v'_q}{v'} = \frac{\delta_p v_p + \dots + \delta_q v_q}{v'} + o(1).$$

Furthermore

$$\frac{\delta_p v'_p + \dots + \delta_q v'_q}{v'} = \frac{\delta_p v_p + \dots + \delta_q v_q}{v} \{1+o(1)\} + o(1)$$

from which it is clear that

$$\limsup_{r \rightarrow \infty} \left| \sum_{s=p}^q \frac{\delta_s v'_s(r)}{v'(r)} \right| > 0.$$

Thus, $U \cup V'$ is an optimally ranked basis.

Next we prove that $|V'|(\mathbf{r}) \neq 0$ for all sufficiently large \mathbf{r} .

We recall that a determinant is a multilinear function of its columns.

If we express

$$V'(\mathbf{r}) = |v'_1, v'_2, \dots, v'_k|(\mathbf{r})$$

where v'_1, v'_2, \dots, v'_k temporarily denote the successive columns of $|V'|(\mathbf{r})$,

then we derive

$$\begin{aligned} |V'|(\mathbf{r}) &= |v_1, v_2, \dots, v_k|(\mathbf{r}) + \sum_{q=1}^j \gamma_{1,q} |u_q, v_2, \dots, v_k|(\mathbf{r}) \\ &= |v_1, v_2, v'_3, \dots, v'_k|(\mathbf{r}) + \sum_{q=1}^j \gamma_{1,q} |u_q, v'_2, \dots, v'_k|(\mathbf{r}) \\ &\quad + \sum_{q=1}^j \gamma_{2,q} |v_1, u_q, v'_3, \dots, v'_k|(\mathbf{r}) \\ &= \dots, \end{aligned}$$

and hence

$$\begin{aligned} |V'|(\mathbf{r}) &= |V|(\mathbf{r}) + \sum_{q=1}^j \gamma_{1,q} |u_q, v'_2, \dots, v'_k|(\mathbf{r}) \\ &\quad + \sum_{q=1}^j \gamma_{2,q} |v_1, u_q, v'_3, \dots, v'_k|(\mathbf{r}) \\ &\quad + \dots \\ &\quad + \sum_{q=1}^j \gamma_{j,q} |v_1, v_2, \dots, v_{k-1}, u_q|(\mathbf{r}). \end{aligned}$$

If we continue this construction until every reference to a primed v is removed, then it is clear that $|V'|(\mathbf{r})$ is equal to the sum of $|V|(\mathbf{r})$ plus a linear combination of determinants formed from $|V|(\mathbf{r})$ by replace-

ment of at least one v by a u . By our assumption that X is j -normal, we conclude

$$(3.2.9) \quad |v'| (r) = |v| (r) \{1+o(1)\} \text{ as } r \rightarrow \infty.$$

Since $|v| (r) \neq 0$ for sufficiently large r , we also have $|v'| (r) \neq 0$ for sufficiently large r .

If one or more of the solutions v'_1, v'_2, \dots, v'_k are replaced in $|v'| (r)$ by subdominant solutions from U , then a construction like the one preceding shows that the modified Casoratian $\tilde{|v'|} (r)$ is dominated by $|v| (r)$ as $r \rightarrow \infty$, since the original optimally ranked basis is j -normal. Consequently, using (3.2.9), we conclude $U \in V'$ is j -normal. A similar argument shows that (3.2.8) is satisfied, and the theorem is proved. \square

3.3 Stability

Let us consider solutions of (3.0.1) of type t , where $0 \leq t < \sigma$.⁺

Let j be the dimension of the subdominant subspace of type t for D .

Let i be a point at which $|U|(i) \neq 0$; see (3.0.7). Suppose D is (i,j) -factorizable; see Definition 1.2.1. Let $A^i = [a_0^i, a_1^i, \dots, a_j^i]$ be a left factor and $B^{i+j} = [b_0^{i+j}, b_1^{i+j}, \dots, b_k^{i+j}]$ the corresponding right factor of D_j^i , so that $D_j^i = A_j^i B^{i+j}$ is an (i,j) -factorization of D .

The next theorem provides a generalization of Theorem 1.3.2, from totally separable linear difference equations to equations that are merely separable. The remark following Theorem 1.3.2 applies here as well. That is, the back substitution stage of the algorithm is stable, at least for sufficiently large r .

Theorem 3.3.1 If $|U|(i) \neq 0$ and $D_j^i = A_j^i B^{i+j}$, then the difference operator B^{i+j} is separable and any solution y of (3.0.1) of type t or less is a sequence of type zero with respect to B^{i+j} .

Proof: Suppose $j = 0$, i.e., type $(y) = 0$ with respect to D . Then $A^i = [a_0^i]$ and $B^i = [b_0^i, b_1^i, \dots, b_\ell^i]$ and we have $D^i = A^i B^i$.

Now let $x \in X$, where X is the optimally ranked basis (3.0.4). Then $Dx = 0$; hence $D^i x^i = A^i B^i x^i = 0$. This last equation is equivalent to

$$a_0(r) \sum_{s=0}^{\ell} b_s(r)x(r+s) = 0, \quad r = i, i+1, \dots.$$

Since a_0^i is free of zeros by the definition of (i,j) - factorizability, we have $B^i x^i = 0$. Thus $\{x^i | x \in X\}$ is an optimally ranked basis for B^i and it follows immediately that type $(y) = 0$ with respect to B^i .

⁺The case $t = \sigma$ corresponds to a maximal solution of (3.0.1). Since the general algorithm reduces to pure forward recurrence in this case, it is stable; see §1.1.

Next suppose j satisfies $0 < j < \ell$. Let $v \in V$, where V is given by (3.0.8). In view of our assumption that $|U|(i) \neq 0$, we may assume that

$$v(i+r) = 0, \quad r = 0, 1, \dots, j-1;$$

compare the proof of Theorem 3.2.1. Since $v \in K(D)$ we have $Dv = 0$. It follows that for each $n \geq i + j + \max(j, k)$ the subsequence $v^{i,n+k-1}$ satisfies the finite boundary value problem

$$D_j^{i,n-j-1} v^{i+j,n-1} = \begin{bmatrix} \dots & \dots & 0 & \dots & \dots \\ -D_{n-\ell,n-j-1} & v^{n,n+k-1} \end{bmatrix};$$

compare (1.2.4). For each n , these boundary value problems are nonsingular and factorizable as shown by Theorem 1.2.1.

Therefore, Theorem 1.2.2 is applicable and we see that $v^{i+j,n-1}$ satisfies

$$B_0^{i+j,n-1} v^{i+j,n-1} = - \begin{bmatrix} \dots & \dots & 0 & \dots & \dots \\ B_k^{n-k,n-1} & v^{n,n+k-1} \end{bmatrix}$$

for each $n \geq i + j + \max(j, k)$; compare (1.2.7) and the remark following the proof of Theorem 1.2.2. The last equation is equivalent to

$$\sum_{s=0}^k b_s(r)v(r+s) = 0, \quad r = i+j, i+j+1, \dots, n-1.$$

Since n may be arbitrarily large, we have

$$\sum_{s=0}^k b_s(r)v(r+s) = 0, \quad r = i+j, i+j+1, \dots$$

or, equivalently, $B^{i+j} v^{i+j} = 0$. Therefore, $\{v^{i+j} | v \in V\}$ is an optimally ranked basis for B^{i+j} . Since y is dominated by every sequence in V ,

it follows that type $(y) = 0$ with respect to B^{i+j} . \square

Now let us turn to the forward elimination stage of the algorithm. This consists of applying a finite number of steps of Gaussian elimination, without pivoting, to the infinite linear system

$$(3.3.1) \quad D_j^i y^{i+j} = g^i - \begin{bmatrix} D_{0,0}^{i,i+j-1} & y^{i,i+j-1} \\ \dots & \dots \\ 0 & 0 \end{bmatrix};$$

compare (1.3.6). Rounding errors introduced in this process are equivalent to perturbations in the original problem. Thus, instead of satisfying (3.3.1) the computed solution \tilde{y}^{i+j} , say, is an exact solution of a system of the form

$$(3.3.2) \quad (D_j^i + E_j^i) \tilde{y}^{i+j} = g^i - \begin{bmatrix} D_{0,0}^{i,i+j-1} & y^{i,i+j-1} \\ \dots & \dots \\ 0 & 0 \end{bmatrix} + e^i$$

where the terms E_j^i and e^i represent the perturbation.

The effect of introducing a single element of $e(r)$, say at $r = s$, is to perturb the true solution y^{i+j} by a linear combination of the solutions of the corresponding homogeneous equation. Because of the boundary conditions we have imposed "at infinity", only the solutions u_1, u_2, \dots, u_j enter from the optimally ranked basis $\{u_1, u_2, \dots, u_j, v_1, v_2, \dots, v_k\}$; compare equation (3.0.3) and Theorem 3.2.1. Subsequently, we have only to consider what happens when $r > s$, and here it will be the multiple of u_j that will be the fastest growing.

The effect of E_j^i , also, can be allowed for by making a perturbation of the right side. To terms of the first order, we have

$$D_j^i \tilde{y}^{i+j} = g^i - \begin{bmatrix} D_{0,0}^{i,i+j-1} & y^{i,i+j-1} \\ \dots & \dots \\ 0 & 0 \end{bmatrix} - E_j^i y^{i+j}.$$

Then the arguments of the preceding paragraph again apply.

Since the wanted solution y is of type t , we conclude that the process of forward elimination is stable in the sense that each perturbation subsequently grows at a rate that does not exceed that of the wanted solution. Of course, there will be a loss of accuracy when an element of E_j^i (or e_j^i) is large compared with the rounding error in the stored value of the corresponding element of D_j^i (or of the right side of (3.3.1)). This may occur, for example, when there is heavy cancellation in the formation of an element that is subsequently used as a pivot. However, the important point is that the effect of each such loss is not magnified in subsequent steps of the algorithm.

An extension of the foregoing discussion shows that if we select a value of j (that is, the number of prescribed initial conditions) that is less than the dimension of the subdominant subspace of the wanted solution, then the forward elimination remains stable. However, the back-substitution is now unstable; compare the proof of Theorem 3.3.1. Similarly if j is too large then the forward elimination is unstable, and the back-substitution is stable. Nevertheless if, in practice, the actual instabilities are weak owing to weak separation of solutions of consecutive types as $r \rightarrow \infty$, then it may be advantageous to select a value of j that differs from the dimension of the subdominant subspace of the wanted solution. This is because the loss of accuracy caused by the instability is offset by a substantial reduction in the value of the terminal point n owing to increased convergence. This modification is illustrated by numerical examples in Chapter 4.

CHAPTER 4. NUMERICAL EXAMPLES

4.0 Introduction

Several numerical examples will be given to illustrate the general algorithm. The examples all involve fourth-order linear difference operators with nonconstant coefficients. Both homogeneous and inhomogeneous difference equations are treated.

The fourth-order operators were produced by using the method described in [14, §12.22], starting with two second-order recurrence relations which have as solutions Bessel functions, modified Bessel functions or associated Legendre functions. The inhomogeneous equations have as particular solutions Anger-Weber functions or Struve functions. The right sides of these equations were produced as a by-product of the method used to produce the fourth-order operators.

The production of the fourth-order difference operators from the corresponding pairs of second-order operators requires only elementary algebraic manipulations. These calculations are rather lengthy, however, and would be extremely difficult (as well as tedious) to complete accurately by hand. Instead, the MACSYMA symbol manipulation system, described in [12], was employed⁺. This procedure had the added advantage of storing the required formulas for the coefficients directly in the computer, ready for numerical evaluation, thereby further reducing the possibility of human error. Similarly, MACSYMA was used to advantage in the production of the right sides for the inhomogeneous examples.

⁺ Development of which is currently supported, in part, by the United States Department of Energy under Contract Number DE-AC02-79ER10357 and by the National Aeronautics and Space Administration under Contract Number NSG1321.

In the case of the Bessel functions $J_r(x)$, $Y_r(x)$ and modified Bessel functions $I_r(x)$, $K_r(x)$ we restrict ourselves to integer order $r \geq 0$ and real argument $x > 0$. Relevant properties of these functions are given in [16, Chapter 9]. For example, for fixed x the functions $J_r(x)$ and $Y_r(x)$ satisfy the linear recurrence relation

$$(4.0.1) \quad y(r-1) - \frac{2r}{x} y(r) + y(r+1) = 0.$$

Similarly, $I_r(x)$ and $(-)^r K_r(x)$ satisfy

$$(4.0.2) \quad y(r-1) - \frac{2r}{x} y(r) - y(r+1) = 0.$$

The Anger-Weber functions $E_r(x)$ satisfy

$$(4.0.3) \quad y(r-1) - \frac{2r}{x} y(r) + y(r+1) = - \frac{2\{1-(-1)^r\}}{\pi x};$$

and the Struve functions $H_r(x)$ satisfy

$$(4.0.4) \quad y(r-1) - \frac{2r}{x} y(r) + y(r+1) = \frac{(\frac{1}{2} x)^r}{\sqrt{\pi} \Gamma(r + \frac{3}{2})}$$

compare (4.0.1) in both cases. Relevant properties of the $E_r(x)$ and $H_r(x)$ appear in Olver's paper [21] where they were used as examples for second-order inhomogeneous equations.

The coefficients of the fourth-order homogeneous recurrence relation obtained from (4.0.1) with $x = x_1$ and (4.0.2) with $x = x_2$ are shown in Figure 6 in the form of FORTRAN statements produced by MACSYMA. In addition, Figure 6 shows the right-hand sides corresponding to $E_r(x_1)$ and $H_r(x_1)$. We designate the difference operator D defined by these

```

ROUTINE(1) = -AO*X1**X2*(X1**2*X2**2-1)*X2**2-R*X2**2+R*X1**2+R
1 2*X1**2
1 DOCTER(2) = 2*AO*X1*X2+X1*(X1**2*X2**2-R*X2**2+R*X2*X1**2
1 2*X1**2)
1 DCOMP(3) = -2*AO*X1*(X1**2*X2**2+2*X1*X2**2-2*X1*X2*X1**2+2*X1*X2*X1**2
1 2*X1**2*X2**2+2*X1*X2*X1**2)
1 DCEFF(4) = -2*AO*X1*(X2-X1)*(X1**2*X2**2-1)*X2**2+R*X2*X1**2
1 2*X1**2*X2**2)
1 DCEPP(5) = AO*X1*X2*(X1**2*X2**2-1)*X2**2+2*X1*X2*X1**2+R*X2*X1**2+R*X1**2
1 X1**2)

CDEFN=0.0D0
IF (ISWR.EQ.0) RETURN
ACDEF(0) = -AO*X1*X2*(X1**2*X2**2-R*X2*X1**2-R*X2*X1**2+R*X2*X1**2+R
1 2*X1**2)
1 ACDEF(1) = 2*AO*X1*(X1**2*X2**2+2*X1*X2**2-1)*X2*X1**2+R*X2*X1**2+R*X2*X1**2+R*X1**2
1 2*X1**2*X2**2)
1 ACDEF(2) = AO*X1*X2*(X1**2*X2**2-R*X2*X1**2-R*X2*X1**2+R*X2*X1**2+R*X1**2
1 X1**2)
1 GO TO (101,102), LDMR
CDEFN=-(1+(-1)**R)*(ACDEF(1)+ACDEF(3))+
1 (1-(-1)**R)*(ACDEF(2))/4.5*07963267943956210001
1 RETURN
101 CDEFN=1.0M/1.6707963267943956210001
DO 102 I=4,R
  CDEFN=Z1*CDEFN/(2*I+1)
102 CDEFN=CDEFN*DCEFF(1)/24+ACDEF(2)+X1*LCOEFF(3)/(2*I+3)
1 2*CDEFN
RETURN

```

Figure 6. Fragment of FORTRAN subroutine for computing coefficients of recurrence relations $\sum_{s=2}^2 d_s(r)y(r+s) = g(r)$ involving the JYIK operator. A0 is an arbitrary scale factor. X1, X2 and R are input parameters. ISWR (=0,1 or 2) selects the homogeneous equation, the inhomogeneous equation having as a particular solution the Anger-Weber function $E_r(x_1)$, or the inhomogeneous equation having as a particular solution the Struve function $H_r(x_1)$.

coefficients the JYIK operator. Since the functions $J_r(x_1)$, $Y_r(x_1)$, $I_r(x_2)$, $(-)^r K_r(x_2)$ are linearly independent, they form a fundamental set of solutions of $Dy = 0$.

Similarly in the case of the associated Legendre functions $P_r^{\mu}(x)$, $Q_r^{\mu}(x)$ we restrict consideration to integer degree $r \geq 0$, integer order $\mu \geq 0$, and real argument $x > 0$. Relevant properties of these functions may be found in [16, Chapter 8]. We use the r -wise linear recurrence relation

$$(4.0.5) \quad (r+\mu)y(r-1) - (2r+1)xy(r) + (r-\mu+1)y(r+1) = 0,$$

valid for fixed x and μ . The coefficients of the fourth-order difference operator D , shown in Figure 7, were produced (using MACSYMA) from (4.0.1) with $x = x_1$ and (4.0.5) with $x = x_2$. Figure 7 also shows the right sides corresponding to $E_r(x_1)$ and $H_r(x_1)$. In this case the functions $J_r(x_1)$, $Y_r(x_1)$, $P_r^{\mu}(x_2)$, $Q_r^{\mu}(x_2)$ form a fundamental set of solutions of $Dy = 0$. Accordingly, we designate D the JYPQ operator.

A FORTRAN program has been written to implement the general algorithm. The user may select either the JYIK operator or the JYPQ operator. He may also elect to solve the homogeneous equation, the inhomogeneous equation having the Anger-Weber function as a solution, or the inhomogeneous equation having the Struve function as a solution. Alternatively, he may provide a subroutine which generates the coefficients of an arbitrary linear difference equation of any order.

The user supplies the program with the starting point i and the **terminal** point m of the desired subsequence of the solution to be computed, the number j of initial values, the subsequence $y^{i,it+j-1}$ of initial values, and the parameters v and ϵ for use in the convergence test (to be described

```

BCCOFF(1) = (-4*R**3+(-4*M-4)*R**2+(5-3*M+3)*R**1*X1**2*X2**2+(8
1   R**2*(2*M+12)*X1**3+(20*M-6)*R**2+(12*M-10)*R-2*X1**2+6*X1-4)*X1*
2   X2+(-3*M**2+4*M-1)*R-4*X1**2-5*M+1*D*X1**2-4*X1**6+(-4*M-4
3   )*R**3+(-M**2-26*M+4)*R**2+(4*X1**3-C*X1**2-3*X1**1)*R**2+(4*X1**3-
4   12*X1**2+C*H*D *R
BCCOFF(2) = ((C*B**3+12*B**2-2*M-3)*X1**3*C*X2**3+C*(-C*B**4+(3*M-24)*
1   B**3+C**M-14)*B**2+(6-2*M)*B**3*X1**2*X2**2+((C**M**2-2)*R*
2   4**M**3-1)*X1**3+(-8*R**3+(4-16*M)*R**4+(-16-8*M)*R**3)*X2+(-4**M**5-
3   2+2*C*M-12)*R**2+(12*X1**2-12*X1*D)*R*X1*X2+(C**M**2-24*M+10)*R**2
4   +(C**M**3-24*M**2+6*M+2)*R-4*X1**2+2*X1**3*X1**2+3*X1**2+3*X1**6+6*X1**3+
5   +(-5*X1**2+32*M-23)*R**4+(-4*X1**3+24*X1**2-24*X1**3+G*X1**2-3
6   2*X1**16)*R**2+(G*X1**3-24*X1**2+16*X1**16)*R*X1
BCCPF(3) = -(16*B**3+4*24*B**3-4*B**2-64*R*X1**2*X2**3+((C**M**3+12
1   R**2+6*M-6)*R+4*M-5)*X1**3+(-3**M**3+5-4*D**2)*C**M**2+(16*M+4)*
2   R**2+(C**M-4)*R*X1**2+X2**2+((-16*X1**4-24*R**2+16*X1**3-24*X1**12)*
3   R**2+(C**M**2-12*X1+10)*R*X1**2+16*X1**3+24*X1**5+24*X1**7+16*X1**9+24*X1**11-4
4   O*X1**4+(-3*X1**3+3*X1**2-4*X1**1)*R**3+(-16*X1**2-4*X1**4-4*X1**6+4*X1**8+4*X1**10+4
5   4*X1**12)*R**2+(4*X1**2-2*X1**4+2*X1**6-2*X1**8+4*X1**10+4*X1**12+4*X1**14+4*X1**16)*R**1
6   R**4-C*X1**2*R**3+(-4*X1**2-2*X1**4-4*X1**6-4*X1**8-12*X1**2-4*X1**4-4*X1**6+4*X1**8+4*X1**10+4*X1**12+4*X1**14+4*X1**16)*R*X1
2   D**2/H1
BCCPF(4) = ((C*B**3+12*B**2-2*M-3)*X1**3*X2**3+C*(-C*B**4-C*B**5+C**M**3+
1   (12-C**M**2+C**M**4+10*M-4)*R**3+6*M-6)*X1**2*X2**2+((C**M**2-2)*R-4*X1**3
2   X2+(-1)*X1**3+(-3*X1**5+(16*X1**4-2*X1**3+(-C**M**2+3*X1**2+3*X1**12)*R**3+(-4*X1**4+4*X1**5+4*X1**6+4*X1**7+3*X1**8+4*X1**9+10)*R**2
3   +9D**2+(-3*X1**3+3*X1**2+16*X1**1)*R**4+((C**M**2-6*X1**2+3*X1**3+6*X1**4+10)*R**3
4   +9D**4+(-3*X1**2+16*X1**3+4*X1**4+12*X1**5+32*X1**6-40*X1**7+16)*R*X1
5   C**M**2-16*X1**2)*R**2+(-3*X1**3+3*X1**2-4*X1**4+16)*R*X1
6   BCCPF(5) = (-4*X1**3+(-4*M-3)*R**2+24*R**1*X1**2*X2**2+(3*X1**4+(-12-M
1   36*X1**5+4*X1**6+10)*R**3+2*X1**2-6*X1**4)*X1*X2+((-4*X1**2+4*X1**4+4*X1**6+4*X1**8-12*X1
2   +12)*R**3+(-4*X1**2+12*X1**3+12*X1**4)*R**2+(4*X1**3-16*X1**5+20*X1**7)
4   *R
CUMPF=0
IF (ISWR.EQ.0) RETURN

```

Figure 7: Fragment of FORTRAN subroutine for computing coefficients of recurrence relations $\sum_{s=-2}^2 d_s(r)y(r+s) = g(r)$ involving the JYQP operator. M(=p), X1, X2 and R are input parameters. ISWR (=0, 1 or 2) selects the homogeneous equation, the inhomogeneous equation having as a particular solution the Anger-Weber function $E_r(x_1)$, or the inhomogeneous equation having as a particular solution the Struve function $H_r(x_1)$.

Figure 7 (Concluded).

below). The program solves the proposed problem by executing three phases. In the initial phase the first $m + l + v - i - j$ rows of the infinite linear system (1.3.6) are stored. Forward elimination without pivoting is performed on these rows. This results in the storing of the first $m + l + v - i - j$ rows of the left and right factors of the (i,j) -factorization $D_j^i = A_j^i B^{i+j}$, and also the subsequence $z^{i+j, m+v}$ of the solution of (1.3.8). The elimination procedure is carried out by modifications, described below, of the subroutines DGBFA and DGBSL which are available in the mathematical software package known as LINPACK; see [8].

The unmodified DGBFA performs forward elimination with partial pivoting on a finite band matrix, ignoring the right side. It produces an upper triangular matrix, a lower triangular matrix and a vector of pivoting information. These results may then be used by the unmodified DGBSL to solve the corresponding linear system with specified right side. Alternatively, the user can direct that DGBSL solve a linear system having as its matrix of coefficients the transpose of the matrix input to DGBFA. Provision has been made in DGBFA to suppress the partial pivoting. When this is done the vector of pivoting information is superfluous and (except for rounding errors) the matrix obtained by premultiplying the upper triangular matrix by the lower triangular matrix is the original band matrix. This factorization is therefore of exactly the kind required by Definition 1.2.1; see equation (1.2.5).

When partial pivoting has been suppressed in DGBFA, DGBSL proceeds, in effect, by forward elimination without pivoting followed by back substitution. Provision has been made in DGBSL to exit after the forward elimination stage and to re-enter at the start of the back substitution stage. Thus the user has the option of examining the result of the

forward elimination stage and perhaps electing to continue it without having to do any back substitution. All of the above modifications of DGBFA and DGBSL, to provide just the tools that are needed in our algorithm, were easily achieved because of the clarity and flexibility of the coding and documentation of LINPACK.

We return to the description of the initial phase. The next computation is the subsequence $w_m^{m+1, m+v}$; compare equations (1.3.16) and (1.3.17) with $s = m$. Since w_m^{m-k+1} satisfies the adjoint equation, it is produced using the forward elimination stage of the (modified) subroutine DGBSL in which the input matrix to DGBFA is automatically transposed. Finally, the initial phase computes

$$(4.0.6) \quad y_r(m) = \sum_{s=m}^{r-1} w_m(s) z(s) / b_o(m)$$

for $r = m+1, m+2, \dots, m+v, m+v+1$ and

$$(4.0.7) \quad \eta_{r,v}(m) = \sum_{s=r}^{r+v-1} w_m(s) z(s) / b_o(m)$$

for $r = m+1$. Equation (4.0.6) is the same as (1.3.18) apart from a change of symbols. Equation (4.0.7) is the approximation to the truncation error

$$\eta_r(m) = y(m) - y_r(m)$$

which is obtained when $y(m)$ is replaced by $y_{r+v}(m)$; compare (1.3.20).

After completion of the initial phase, the program proceeds to the iteration phase. One additional row of the infinite linear system (1.3.6) is stored. Since the previously determined part of the (i,j) -factorization has replaced the corresponding rows of the original matrix in the computer

storage, these rows are no longer available to extend the forward elimination stage. Therefore, extension of the forward elimination is accomplished by an alternative procedure that requires the solutions of two triangular linear systems of orders j and k . This is achieved using DTRSL, a LINPACK subroutine that solves triangular systems. Next, the formerly obtained subsequences of the solutions z^{i+j} and w_m^{m+1} are extended to include the next higher term. Then the series (4.0.6) and (4.0.7) are summed with the new value of r , and the following convergence test is applied: If

$$(4.0.8) \quad |n_{r,v}(m)| \leq \varepsilon |y_{r+v}(m)|$$

then the program proceeds to the third and final phase; otherwise, the iteration phase is repeated and the convergence test re-applied.

The final phase of the program is the back substitution stage of the algorithm. Let n denote the value of r which satisfies (4.0.8). The back substitution is begun at this point, using the (modified) subroutine DGBSL. This yields the computed solution $y_n^{i+j,n-1}$ of (1.3.7) and also the desired approximation of $y(r)$ over the range $r = i+j, i+j+1, \dots, m$.

In all of the examples presented in this thesis the chosen termination criterion corresponds to an estimated relative error of 10^{-10} in the final term $y_n(m)$ of the requested sequence of values of the function. Thus $\varepsilon = 10^{-10}$ in equation (4.0.8). For the values of x_1 and x_2 selected in the examples the solutions are reasonably well-separated as $r \rightarrow \infty$, hence we take the "testing parameter" v in equation (4.0.7) to be 1 throughout. Initial values were entered to 10 significant figures in every case. All computations were performed internally to 18 figures

and the output printed to 10 figures. For each example we present (i) the solution $z(i+j), z(i+j+1), \dots, z(n)$ of equation (1.3.8) (the forward elimination stage); (ii) the solution $w_m(m+1), w_m(m+2), \dots, w_m(n)$ of (1.3.16), (1.3.17) (the forward recurrence of the adjoint equation for use in the termination criterion); (iii) the sequence $y_{m+1}(m), y_{m+2}(m), \dots, y_{n+v}(m)$ of approximants to the value $y(m)$ (see equation (4.0.6)); (iv) the sequence $\eta_{m+1,1}(m), \eta_{m+2,1}(m), \dots, \eta_{n,1}(m)$ of estimates of the truncation error incurred in accepting the terminal points of $m+1, m+2, \dots, n$ respectively (see equation (4.0.7)); (v) the sequence $y_n(n-1), y_n(n-2), \dots, y_n(i+j)$ derived from equation (1.3.7) (the back substitution stage).

The initial values that were used in the boundary value problem actually solved in each example are shown in parentheses in the tables; see, for example, Table 5. The terminal values (which are all zero) are also shown. Forward recurrence of the j -th order difference equation $A^i z^i = g^i$ starting from the given initial values produces a solution of (1.3.8) (the algorithm produces this solution by forward elimination). Similarly, backward recurrence of the k -th order difference equation $B^{i+j} y_n^{i+j} = z^{i+j}$ starting from the terminal values (which are all zero) produces the corresponding solution of the boundary value problem; this is the back substitution stage. Data is given in the tables only for selected values of r . All of the values of r that were used beyond $r = m$ are included, however. These values illustrate the termination procedure.

It should be noted that when $j = 4$ the method as implemented by the FORTRAN program is equivalent to, but not identical with, forward recurrence of the given difference equation. See, for example, Table 8. This is because DGBFA, the LINPACK matrix factorization subroutine, scales the linear system in such a way

that the principal diagonal of the lower triangular factor is a sequence of ones. Therefore, the result of the forward elimination stage differs from the result of the back substitution stage (unless the trailing coefficients of the original linear difference operator are all equal to one). Since there is no question of truncation error when $j = 4$, the sequences $w_m(r)$, $y_r(m)$ and $\eta_{r,1}(m)$, $r = m+1, m+2, \dots$, are not defined.

It has been observed that Olver's original second-order method is likely to cause overflow in most computers. This same likelihood of overflow exists for the extended algorithm. The examples to be presented in this chapter were computed using double precision on a Univac 1108. Double precision was used not for higher precision but for wider exponent range.

The double precision range extends to 10^{308} whereas the largest number generated in the examples was somewhat greater than 10^{200} . Van der Cruyssen [25] gives a rescaled version of the second-order algorithm, based on LU decomposition, that greatly reduces the incidence of overflow or underflow. An equivalent procedure for overcoming this difficulty, based directly on the use of ratios of consecutive terms in each solution, is described by Aggarwal and Burgmeier [1]. In the present writer's view, however, it is preferable (if possible) to avoid mathematical modifications of algorithms simply to suit arbitrary limitations of existing computers. A software alternative to rescaling is proposed in [24], in which a full integer storage location is allocated to the exponent of each floating-point number. A FORTRAN implementation of this proposal is available; see [11].

4.1 Examples Involving the Homogeneous JYIK Equation

From the asymptotic forms

$$(4.1.1) \quad J_r(x) \sim \frac{1}{(2\pi r)^{1/2}} \left(\frac{ex}{2r}\right)^r, \quad Y_r(x) \sim -\left(\frac{2}{\pi r}\right)^{1/2} \left(\frac{ex}{2r}\right)^{-r}$$

and

$$(4.1.2) \quad I_r(x) \sim \frac{1}{(2\pi r)^{1/2}} \left(\frac{ex}{2r}\right)^r, \quad (-)^r K_r(x) \sim (-)^r \left(\frac{\pi}{2r}\right)^{1/2} \left(\frac{ex}{2r}\right)^{-r}$$

which are valid for fixed x and large r , we obtain

$$(4.1.3) \quad J_r(x_1) < I_r(x_2) < (-)^r K_r(x_2) < Y_r(x_1)$$

when $x_1 < x_2$ and

$$(4.1.4) \quad I_r(x_2) < J_r(x_1) < Y_r(x_1) < (-)^r K_r(x_2)$$

when $x_2 < x_1$, where in (4.1.3) and (4.1.4) the symbol $<$ is used to indicate the order of dominance of the functions as $r \rightarrow \infty$. Thus when $x_1 \neq x_2$ the JYIK operator is totally separable. As with any totally separable operator of order 4, there are four distinct types of solution and the dimension of the subdominant subspace of type s is s . This means that $j = s$ is the proper number of initial values to specify for a solution of type s .

When all four Bessel functions have the same argument, from (4.1.1) and (4.1.2) we have

$$(4.1.5) \quad J_r(x) \sim I_r(x), \quad (-)^r K_r(x) \sim (-)^{r+1} \frac{1}{2} \pi Y_r(x), \quad r \rightarrow \infty.$$

The first of these relations suggests that we define the further solution

$$(4.1.6) \quad A_r(x) = J_r(x) - I_r(x) .$$

Then it may be verified that

$$(4.1.7) \quad A_r(x) \sim -\frac{1}{2} \frac{x^2}{r+1} \left\{ \begin{array}{l} I_r(x) \\ J_r(x) \end{array} \right\}, \quad r \rightarrow \infty$$

where either function in the braces on the right may be chosen. Furthermore, since there is no nonzero linear combination of $(-)^r K_r(x)$ and $Y_r(x)$ that reduces to a solution of lower type, we conclude that there are exactly three distinct types of solution. Symbolically, we have

$$(4.1.8) \quad A_r(x) < \left\{ \begin{array}{l} I_r(x) \\ J_r(x) \end{array} \right\} < \left\{ \begin{array}{l} (-)^r K_r(x) \\ Y_r(x) \end{array} \right\}$$

where in each pair of braces either function may be chosen. Thus when $x_1 = x_2$ the JYIK operator is separable but not totally separable. The dimensions of the subdominant subspaces of types 1, 2 and 3 are 1, 2 and 4 respectively.

Although $A_r(x)$ has type 1, it is separated only by a relative factor of $1/r$ from solutions of type 2. Let $y_n(r)$ denote the approximation to $A_r(x)$ that is obtained from the algorithm with terminal point n . Then, suppressing the arguments of $A_r(x)$ and all the Bessel functions and using equation (3.2.6), we have⁺

$$y_n(r) \sim A_r - \beta_1(n) I_r - \beta_2(n) Y_r - \beta_3(n) (-)^r K_r, \quad n \rightarrow \infty$$

where

⁺ Since $I_r(x)$, $Y_r(x)$ and $K_r(x)$ are not normalized as required for the validity of (3.2.6), we have only the weaker statement given here.

$$\beta_1(n) = \begin{vmatrix} A_n & Y_n & (-)^n K_n \\ A_{n+1} & Y_{n+1} & (-)^{n+1} K_{n+1} \\ A_{n+2} & Y_{n+2} & (-)^{n+2} K_{n+2} \end{vmatrix} \Bigg/ \begin{vmatrix} I_n & Y_n & (-)^n K_n \\ I_{n+1} & Y_{n+1} & (-)^{n+1} K_{n+1} \\ I_{n+2} & Y_{n+2} & (-)^{n+2} K_{n+2} \end{vmatrix},$$

$$\beta_2(n) = \begin{vmatrix} I_n & A_n & (-)^n K_n \\ I_{n+1} & A_{n+1} & (-)^{n+1} K_{n+1} \\ I_{n+2} & A_{n+2} & (-)^{n+2} K_{n+2} \end{vmatrix} \Bigg/ \begin{vmatrix} I_n & Y_n & (-)^n K_n \\ I_{n+1} & Y_{n+1} & (-)^{n+1} K_{n+1} \\ I_{n+2} & Y_{n+2} & (-)^{n+2} K_{n+2} \end{vmatrix},$$

and

$$\beta_3(n) = \begin{vmatrix} I_n & Y_n & A_n \\ I_{n+1} & Y_{n+1} & A_{n+1} \\ I_{n+2} & Y_{n+2} & A_{n+2} \end{vmatrix} \Bigg/ \begin{vmatrix} I_n & Y_n & (-)^n K_n \\ I_{n+1} & Y_{n+1} & (-)^{n+1} K_{n+1} \\ I_{n+2} & Y_{n+2} & (-)^{n+2} K_{n+2} \end{vmatrix}.$$

In consequence of (4.1.1), (4.1.2), and (4.1.7) we see that

$\beta_1(n) \sim -x^2(2n)^{-1}$ for large n , whereas $\beta_2(n)$ and $\beta_3(n)$ approach zero much more rapidly. Thus we expect that the rate of convergence of the algorithm will be approximately as $1/n$. This has been confirmed in numerical tests. Since the separation of $A_r(x)$ from solutions of type two is so weak, it is usually advantageous to regard $A_r(x)$ as belonging to the same type as $J_r(x)$ and $I_r(x)$; this means that we specify two initial conditions rather than one. This modification improves convergence; in consequence the value of the terminal point n is reduced very considerably. This gain is offset to a minor extent by some loss in accuracy in the values of $A_r(x)$ for higher values of r ; the extent of this loss is no more than the cancellation that takes place between the corresponding values of $I_r(x)$ and $J_r(x)$ in Eq. (4.1.6).

Example 4.1.1 In this example we compute the solutions $J_r(1)$, $I_r(10)$, $(-)^r K_r(10)$ and $Y_r(1)$ of the homogeneous equation defined in Figure 6, with $x_1 = 1$, $x_2 = 10$, for $r = 0, 1, 2, \dots, 100$. Since each of these solutions is monotonic (in magnitude), the algorithm will be numerically stable provided that the proper number j of initial values is used in each case. This number is $j = 1$ for $J_r(1)$, $j = 2$ for $I_r(10)$, $j = 3$ for $(-)^r K_r(10)$ and $j = 4$ for $Y_r(1)$; compare (4.1.3).

Table 1 gives the numerical coefficients of the JYIK operator with $x_1 = 1$ and $x_2 = 10$ for $r = 0, 1, 2, \dots, 109$. Tables 2, 3 and 4 give the numerical coefficients of the $(0, j)$ -factorizations of this JYIK operator for $j = 1, 2$, and 3. (We do not present the $(0, 4)$ -factorization.) Since the (i, j) -factorization $D_j^i = A_j^i B^{i+j}$ produced by the test program has the principal diagonal of A_j^i equal to $(1, 1, 1, \dots)$, we do not include values of $a_j(r)$.

Tables 5 through 8 give the results of the computation of the four desired solutions $J_r(1)$, $I_r(10)$, $(-)^r K_r(10)$ and $Y_r(1)$ for selected values of r ; see §4.0 for a full description of the tables. Initial values were taken from Tables 9.4 and 9.11 in [16]; they are given in parentheses. These tables also supply values of the four Bessel functions for certain other values of r in the range $0 \leq r \leq 100$. These entries all agree with our computed results, except for the tenth significant figure in some instances. This slight discrepancy is explained by the fact that the starting values that were used inevitably have an error of several units in the eleventh significant figure. Incidentally, because in this (and other examples) we have not attempted rigorous control of individual rounding errors, a guaranteed error bound cannot be given.

Nevertheless, the stability of the algorithm for all solutions in this example is firmly demonstrated.

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
0	-4926000000000004	-3663000000000003	-2000000000000002	-9994000000000001	-4946000000000000
1	.1024000000000005	.4335200000000005	.9922000000000005	.4946000000000004	.1035000000000005
2	.4567200000000005	.1210500000000005	.2175000000000005	.4642000000000005	.1823000000000005
3	.2503000000000005	.4424000000000005	.4224000000000006	.4644000000000006	.2350000000000006
4	.4091900000000005	.1660000000000006	.4076500000000006	.5381320000000006	.4945000000000006
5	.5413000000000005	.2163000000000006	.4076500000000006	.6381320000000006	.5424000000000006
6	.7022000000000005	.1030112000000006	.1654912000000006	.1266400000000006	.7623000000000006
7	.5040000000000005	.2000000000000006	.2515600000000006	.1746114000000006	.6350000000000006
8	.1072000000000006	.2163200000000006	.3266200000000006	.2352144000000006	.1079000000000006
9	.1286000000000006	.2326600000000006	.4676200000000006	.2036312000000006	.9940000000000006
10	.1524000000000006	.3871400000000006	.2000000000000006	.2036312000000006	.4296380000000006
11	.1721000000000006	.4222515200000006	.1144937200000006	.1266400000000006	.1534400000000006
12	.2062000000000006	.3915140000000006	.1532843600000006	.2000000000000006	.1791000000000006
13	.2286200000000006	.2163200000000006	.1933816000000006	.2900000000000006	.6260520000000006
14	.2624000000000006	.3056100000000006	.2652161600000006	.2036312000000006	.2366000000000006
15	.3010000000000006	.1062100000000006	.3236000000000006	.1642400000000006	.3664000000000006
16	.3475000000000006	.1268232920000006	.4122232600000006	.1933816000000006	.5461942000000006
17	.3762000000000006	.1435162350000006	.5112039600000006	.1215460000000006	.6346400000000006
18	.4143000000000006	.1235643000000006	.6050120300000006	.1413416000000006	.4742000000000006
19	.4562000000000006	.2007265290000006	.7662100000000006	.1642400000000006	.4146200000000006
20	.5040000000000006	.2329490500000006	.9232000000000006	.1882492000000006	.4563000000000006
21	.5484000000000006	.2639900320000006	.1103494920400006	.1268232600000006	.4992400000000006
22	.5900000000000006	.3000000000000006	.1268232600000006	.2454340000000006	.5454400000000006
23	.6326000000000006	.3396100000000006	.1541394000000006	.2275400000000006	.5936000000000006
24	.6752000000000006	.3700000000000006	.1822300000000006	.3142300000000006	.6345000000000006
25	.7178000000000006	.4000000000000006	.2028153120400006	.3492323400000006	.6743000000000006
26	.7604000000000006	.4322163020000006	.2432323418200006	.3910161400000006	.7144400000000006
27	.8030000000000006	.4766213240000006	.2879552600000006	.4392152000000006	.7652000000000006
28	.8456000000000006	.5229222000000006	.3199630000000006	.5009400000000006	.8162000000000006
29	.8882000000000006	.5600000000000006	.3416490300000006	.5629240000000006	.8672000000000006
30	.9308000000000006	.6000000000000006	.3649354000000006	.6329240000000006	.9182000000000006
31	.9734000000000006	.6426000000000006	.3876230000000006	.6992152000000006	.9691000000000006
32	.1016000000000006	.6900000000000006	.4149400000000006	.7592141666666666	.1044400000000006
33	.1061600000000006	.7400000000000006	.4416490300000006	.8201222222222222	.1104300000000006
34	.1112000000000006	.7900000000000006	.4646400000000006	.8801201200000006	.1174600000000006
35	.1169000000000006	.8400000000000006	.4904311120000006	.9430000000000006	.1220200000000006
36	.1236100000000006	.8900000000000006	.5163400000000006	.9994000000000006	.1292000000000006

Table 1. Numerical coefficients of the JYIK operator with $x_1 = 1$ and $x_2 = 10$.

Table 1 (Continued)

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
37	-1543400000+000	-12902604000+000	.9143331400+000	-14661399000+000	-10556676000+000
38	-1622600000+000	-13921601000+000	.10124656000+000	-1390420000+000	-1600+000000000000
39	-1703740000+000	-1499304400+000	.1112530460+000	-1623660000+000	-1600+000000000000
40	-1736940000+000	-161179300+000	.1230742632+010	-1623660000+000	-1600+000000000000
41	-1823039999+000	-1729779920+000	.1352386532+010	-15244300+000	-1600+000000000000
42	-1959300000+000	-1863381200+000	.1482616030+010	-1640230000+000	-1600+000000000000
43	-2048300000+000	-1902732400+000	.1622143200+010	-1623560000+000	-1600+000000000000
44	-2149300000+000	-2112964200+000	.1721295320+010	-1732079300+000	-1600+000000000000
45	-2230440000+000	-2259207230+000	.1920500322+010	-1834442320+000	-1600+000000000000
46	-2327420000+000	-2406593230+000	.21602061792+010	-1960300300+000	-1600+000000000000
47	-2424540000+000	-2560250000+000	.2230051020+010	-2094760000+000	-1600+000000000000
48	-2523500000+000	-272031000+000	.2422960000+010	-2222525000+000	-1600+000000000000
49	-2624400000+000	-2836906000+000	.2676346130+010	-2362014000+000	-1600+000000000000
50	-2722740000+000	-3000165620+000	.2895142512+010	-24032271920+000	-1600+000000000000
51	-2822340000+000	-315621239600+000	.3122232322+010	-2451036360+000	-1600+000000000000
52	-2922930000+000	-3242201200+000	.3362422764+010	-25040240260+000	-1600+000000000000
53	-3023500000+000	-33421239600+000	.3562124350+010	-25532000+000	-1600+000000000000
54	-3124400000+000	-343924400+000	.3762423264+010	-26026000+000	-1600+000000000000
55	-3225300000+000	-353562120+000	.3963422214+010	-265211242214+000	-1600+000000000000
56	-3326200000+000	-3635362120+000	.416215252214+010	-270211242214+000	-1600+000000000000
57	-3427100000+000	-3735362120+000	.43656342900+010	-275211242214+000	-1600+000000000000
58	-3528000000+000	-3835362120+000	.45685362120+000	-280211242214+000	-1600+000000000000
59	-3628900000+000	-3935362120+000	.4771835362120+000	-285211242214+000	-1600+000000000000
60	-3729800000+000	-4035362120+000	.4975135362120+000	-290211242214+000	-1600+000000000000
61	-3830700000+000	-4135362120+000	.5178435362120+000	-295211242214+000	-1600+000000000000
62	-3931600000+000	-4235362120+000	.5381735362120+000	-300211242214+000	-1600+000000000000
63	-4032500000+000	-4335362120+000	.5585035362120+000	-305211242214+000	-1600+000000000000
64	-4133400000+000	-4435362120+000	.5788335362120+000	-310211242214+000	-1600+000000000000
65	-4234300000+000	-4535362120+000	.5991635362120+000	-315211242214+000	-1600+000000000000
66	-4335200000+000	-4635362120+000	.6195035362120+000	-320211242214+000	-1600+000000000000
67	-4436100000+000	-4735362120+000	.6398335362120+000	-325211242214+000	-1600+000000000000
68	-4537000000+000	-4835362120+000	.6601635362120+000	-330211242214+000	-1600+000000000000
69	-4637900000+000	-4935362120+000	.6805035362120+000	-335211242214+000	-1600+000000000000
70	-4738800000+000	-5035362120+000	.7008335362120+000	-340211242214+000	-1600+000000000000
71	-4839700000+000	-5135362120+000	.7211635362120+000	-345211242214+000	-1600+000000000000
72	-4940600000+000	-5235362120+000	.7415035362120+000	-350211242214+000	-1600+000000000000
73	-5041500000+000	-5335362120+000	.7618335362120+000	-355211242214+000	-1600+000000000000
74	-5142400000+000	-5435362120+000	.7821635362120+000	-360211242214+000	-1600+000000000000
75	-5243300000+000	-5535362120+000	.8025035362120+000	-365211242214+000	-1600+000000000000
76	-5344200000+000	-5635362120+000	.8228335362120+000	-370211242214+000	-1600+000000000000
77	-5445100000+000	-5735362120+000	.8431635362120+000	-375211242214+000	-1600+000000000000
78	-5546000000+000	-5835362120+000	.8635035362120+000	-380211242214+000	-1600+000000000000
79	-5646900000+000	-5935362120+000	.8838335362120+000	-385211242214+000	-1600+000000000000
80	-5747800000+000	-6035362120+000	.9041635362120+000	-390211242214+000	-1600+000000000000
81	-5848700000+000	-6135362120+000	.9245035362120+000	-395211242214+000	-1600+000000000000
82	-5949600000+000	-6235362120+000	.9448335362120+000	-400211242214+000	-1600+000000000000
83	-6050500000+000	-6335362120+000	.9651635362120+000	-405211242214+000	-1600+000000000000
84	-6151400000+000	-6435362120+000	.9855035362120+000	-410211242214+000	-1600+000000000000
85	-6252300000+000	-6535362120+000	.10058335362120+000	-415211242214+000	-1600+000000000000
86	-6353200000+000	-6635362120+000	.10261635362120+000	-420211242214+000	-1600+000000000000
87	-6454100000+000	-6735362120+000	.10465035362120+000	-425211242214+000	-1600+000000000000
88	-6555000000+000	-6835362120+000	.10668335362120+000	-430211242214+000	-1600+000000000000
89	-6655900000+000	-6935362120+000	.10871635362120+000	-435211242214+000	-1600+000000000000
90	-6756800000+000	-7035362120+000	.11075035362120+000	-440211242214+000	-1600+000000000000
91	-6857700000+000	-7135362120+000	.11278335362120+000	-445211242214+000	-1600+000000000000
92	-6958600000+000	-7235362120+000	.11481635362120+000	-450211242214+000	-1600+000000000000
93	-7059500000+000	-7335362120+000	.11685035362120+000	-455211242214+000	-1600+000000000000
94	-7160400000+000	-7435362120+000	.11888335362120+000	-460211242214+000	-1600+000000000000
95	-7261300000+000	-7535362120+000	.12091635362120+000	-465211242214+000	-1600+000000000000
96	-7362200000+000	-7635362120+000	.12295035362120+000	-470211242214+000	-1600+000000000000
97	-7463100000+000	-7735362120+000	.12498335362120+000	-475211242214+000	-1600+000000000000
98	-7564000000+000	-7835362120+000	.12701635362120+000	-480211242214+000	-1600+000000000000
99	-7664900000+000	-7935362120+000	.12905035362120+000	-485211242214+000	-1600+000000000000
100	-7765800000+000	-8035362120+000	.13108335362120+000	-490211242214+000	-1600+000000000000
101	-7866700000+000	-8135362120+000	.13311635362120+000	-495211242214+000	-1600+000000000000
102	-7967600000+000	-8235362120+000	.13515035362120+000	-500211242214+000	-1600+000000000000
103	-8068500000+000	-8335362120+000	.13718335362120+000	-505211242214+000	-1600+000000000000
104	-8169400000+000	-8435362120+000	.13921635362120+000	-510211242214+000	-1600+000000000000
105	-8260300000+000	-8535362120+000	.14125035362120+000	-515211242214+000	-1600+000000000000
106	-8361200000+000	-8635362120+000	.14328335362120+000	-520211242214+000	-1600+000000000000
107	-8462100000+000	-8735362120+000	.14531635362120+000	-525211242214+000	-1600+000000000000
108	-8563000000+000	-8835362120+000	.14735035362120+000	-530211242214+000	-1600+000000000000
109	-8663900000+000	-8935362120+000	.14938335362120+000	-535211242214+000	-1600+000000000000
110	-8764800000+000	-9035362120+000	.15141635362120+000	-540211242214+000	-1600+000000000000
111	-8865700000+000	-9135362120+000	.15345035362120+000	-545211242214+000	-1600+000000000000
112	-8966600000+000	-9235362120+000	.15548335362120+000	-550211242214+000	-1600+000000000000
113	-9067500000+000	-9335362120+000	.15751635362120+000	-555211242214+000	-1600+000000000000
114	-9168400000+000	-9435362120+000	.15955035362120+000	-560211242214+000	-1600+000000000000
115	-9269300000+000	-9535362120+000	.16158335362120+000	-565211242214+000	-1600+000000000000
116	-9360200000+000	-9635362120+000	.16361635362120+000	-570211242214+000	-1600+000000000000
117	-9461100000+000	-9735362120+000	.16565035362120+000	-575211242214+000	-1600+000000000000
118	-9562000000+000	-9835362120+000	.16768335362120+000	-580211242214+000	-1600+000000000000
119	-9662900000+000	-9935362120+000	.16971635362120+000	-585211242214+000	-1600+000000000000
120	-9763800000+000	-10035362120+000	.17175035362120+000	-590211242214+000	-1600+000000000000
121	-9864700000+000	-10135362120+000	.17378335362120+000	-595211242214+000	-1600+000000000000
122	-9965600000+000	-10235362120+000	.17581635362120+000	-600211242214+000	-1600+000000000000
123	-10066500000+000	-10335362120+000	.17785035362120+000	-605211242214+000	-1600+000000000000
124	-10167400000+000	-10435362120+000	.17988335362120+000	-610211242214+000	-1600+000000000000
125	-10268300000+000	-10535362120+000	.18191635362120+000	-615211242214+000	-1600+000000000000
126	-10369200000+000	-10635362120+000	.18395035362120+000	-620211242214+000	-1600+000000000000
127	-10460100000+000	-10735362120+000	.18598335362120+000	-625211242214+000	-1600+000000000000
128	-10561000000+000	-10835362120+000	.18801635362120+000	-630211242214+000	-1600+000000000000
129	-10661900000+000	-10935362120+000	.19005035362120+000	-635211242214+000	-1600+000000000000
130	-10762800000+000	-11035362120+000	.19208335362120+000	-640211242214+000	-1600+000000000000
131	-10863700000+000	-11135362120+000	.19411635362120+000	-645211242214+000	-1600+000000000000
132	-10964600000+000	-11235362120+000	.19615035362120+000	-650211242214+000	-1600+000000000000
133	-11065500000+000	-11335362120+000	.19818335362120+000	-655211242214+000	-1600+000000000000
134	-11166400000+000	-11435362120+000	.20021635362120+000	-660211242214+000	-1600+000000000000
135	-11267300000+000	-11535362120+000	.20225035362120+000	-665211242214+000	-1600+000000000000
136	-11368200000+000	-11635362120+000	.20428335362120+000	-670211242214+000	-1600+000000000000
137	-11469100000+000	-11735362120+000	.20631635362120+000	-675211242214+000	-1600+000000000000
138	-11560000000+000	-11835362120+000	.20835035362120+000	-680211242214+000	-1600+000000000000
139	-11660900000+000	-11935362120+000	.21038335362120+000	-685211242214+000	-1600+000000000000
140	-11761800000+000	-12035362120+000	.21241635362120+000	-690211242214+000	-1600+000000000000
141	-11862700000+000	-12135362120+000	.21445035362120+000	-695211242214+000	-1600+000000000000
142	-1196				

Table 1 (Concluded).

r	$a_0(r)$	$b_0(r+1)$	$b_1(r+1)$	$b_2(r+1)$
0	- .33643900000+004	- .70920000000+003	- .93300000000+003	- .49420000000+003
1	- .12355491509+001	- .27753638103+005	- .3613791232+005	- .49420000000+003
2	- .4018242676+000	- .95031093774+005	- .9734938762+005	- .02400000000+005
3	- .1107273396+005	- .24746328297+006	- .20204996385+006	- .10000000000+005
4	- .22671459436+006	- .50477699492+006	- .3600400644+006	- .20000000000+005
5	- .441226790+006	- .9465595062+006	- .5822549274+006	- .40000000000+005
6	- .1369223905+000	- .1616133652+007	- .8793217654+006	- .54445650000+005
7	- .10038241250+000	- .9773299531+006	- .1261932535+007	- .70230000000+005
8	- .9014355321-001	- .1462495674+007	- .12407273056+007	- .81100000000+005
9	- .269342654-0.21	- .1934593599+007	- .200+000	- .900+0000000+005
10	- .6703216639-001	- .2686494554+007	- .37959492489+007	- .10790200000+005
11	- .5934645966-001	- .3362026399+007	- .3029900015+007	- .12940000000+005
12	- .45301560369-001	- .4291219604+007	- .31615370939+008	- .53445000000+005
13	- .4321523250-001	- .5369391931+007	- .4383261931+008	- .1794590394+006
14	- .4406121549-001	- .6614272541+007	- .69526251+008	- .26694000000+006
15	- .4055611644-001	- .259575252525+000	- .723336363525+000	- .90316666666+006
16	- .3756496551-001	- .96460992129+007	- .8635134129+007	- .26822000000+006
17	- .3497873720-001	- .1146613639+008	- .1031877339+008	- .30194900000+006
18	- .32722944491-001	- .1349310973+008	- .12144279923+008	- .32732000000+006
19	- .3083703279-001	- .16249323279+008	- .14173938789+008	- .37240000000+006
20	- .28976463213-001	- .18242127279+008	- .16417000000+008	- .41430000000+006
21	- .2740579405-001	- .20663391293+008	- .143324564031+008	- .45630000000+006
22	- .2509515276-001	- .23693205633+008	- .214632364633+008	- .49994660000+006
23	- .2472166621-001	- .27226354065+008	- .246363665+008	- .54545454545+006
24	- .2366635290-001	- .30032404502+008	- .27424343502+008	- .59000000000+006
25	- .2261353435-001	- .34663344644+008	- .300+0000000+008	- .64256600000+006
26	- .21856323631-001	- .38365036391+008	- .31213949437+008	- .69393930000+006
27	- .203665720320-001	- .43330927204+008	- .3210543431+009	- .74734600000+006
28	- .19260954920-001	- .4832503252+008	- .32441242951+009	- .80000000000+006
29	- .18996365430-001	- .53032503252+008	- .32441242951+009	- .84799126524+006
30	- .18393810414-001	- .58322492249+008	- .32441242951+009	- .8926069214+006
31	- .17742112-001	- .6300+0000000+008	- .32441242951+009	- .9500+0000000+006
32	- .16557252639-001	- .680+0000000+008	- .32441242951+009	- .9900+0000000+006
33	- .16004934421-001	- .730+0000000+008	- .32441242951+009	- .9900+0000000+006
34	- .15350563942-001	- .780+0000000+008	- .32441242951+009	- .9900+0000000+006
35	- .1500332042-001	- .830+0000000+008	- .32441242951+009	- .9900+0000000+006
36	- .1456424499-001	- .880+0000000+008	- .32441242951+009	- .9900+0000000+006

Table 2. Numerical coefficients of the $(0,1)$ -factorization of the JYlk operator with $x_1 = 1$ and $x_2 = 10$.

r	a ₀ (r)	b ₀ (r+1)	b ₁ (r+1)	b ₂ (r+1)
1	-14322	-	-	-
2	-109330939-001	-	-	-
3	-131123244-001	-	-	-
4	-1456562381-001	-	-	-
5	-1625142432-001	-	-	-
6	-205683238-001	-	-	-
7	-2422353625-001	-	-	-
8	-286950321-010	-	-	-
9	-336392052-010	-	-	-
10	-394214232-010	-	-	-
11	-460299548-010	-	-	-
12	-5359255-010	-	-	-
13	-619113257-010	-	-	-
14	-712854956-010	-	-	-
15	-8165352-010	-	-	-
16	-92161430-010	-	-	-
17	-10325252-010	-	-	-
18	-1145656238-001	-	-	-
19	-126223238-001	-	-	-
20	-13790434-001	-	-	-
21	-15056238-001	-	-	-
22	-16325252-001	-	-	-
23	-17590434-001	-	-	-
24	-18856238-001	-	-	-
25	-201223238-001	-	-	-
26	-213890434-001	-	-	-
27	-226556238-001	-	-	-
28	-239223238-001	-	-	-
29	-251890434-001	-	-	-
30	-264556238-001	-	-	-
31	-277223238-001	-	-	-
32	-290890434-001	-	-	-
33	-303556238-001	-	-	-
34	-316223238-001	-	-	-
35	-328890434-001	-	-	-
36	-341556238-001	-	-	-
37	-354223238-001	-	-	-
38	-366890434-001	-	-	-
39	-379556238-001	-	-	-
40	-392223238-001	-	-	-
41	-404890434-001	-	-	-
42	-417556238-001	-	-	-
43	-430223238-001	-	-	-
44	-442890434-001	-	-	-
45	-455556238-001	-	-	-
46	-468223238-001	-	-	-
47	-480890434-001	-	-	-
48	-493556238-001	-	-	-
49	-506223238-001	-	-	-
50	-518890434-001	-	-	-
51	-531556238-001	-	-	-
52	-544223238-001	-	-	-
53	-556890434-001	-	-	-
54	-569556238-001	-	-	-
55	-582223238-001	-	-	-
56	-594890434-001	-	-	-
57	-607556238-001	-	-	-
58	-620223238-001	-	-	-
59	-632890434-001	-	-	-
60	-645556238-001	-	-	-
61	-658223238-001	-	-	-
62	-670890434-001	-	-	-
63	-683556238-001	-	-	-
64	-696223238-001	-	-	-
65	-708890434-001	-	-	-
66	-721556238-001	-	-	-
67	-734223238-001	-	-	-
68	-746890434-001	-	-	-
69	-759556238-001	-	-	-
70	-772223238-001	-	-	-
71	-784890434-001	-	-	-
72	-797556238-001	-	-	-
73	-810223238-001	-	-	-
74	-822890434-001	-	-	-
75	-835556238-001	-	-	-
76	-848223238-001	-	-	-
77	-860890434-001	-	-	-
78	-873556238-001	-	-	-
79	-886223238-001	-	-	-
80	-898890434-001	-	-	-
81	-911556238-001	-	-	-
82	-924223238-001	-	-	-
83	-936890434-001	-	-	-
84	-949556238-001	-	-	-
85	-962223238-001	-	-	-
86	-974890434-001	-	-	-
87	-987556238-001	-	-	-
88	-990223238-001	-	-	-
89	-1002890434-001	-	-	-
90	-101556238-001	-	-	-
91	-1028223238-001	-	-	-
92	-1040890434-001	-	-	-
93	-1053556238-001	-	-	-
94	-1066223238-001	-	-	-
95	-1078890434-001	-	-	-
96	-1091556238-001	-	-	-
97	-1104223238-001	-	-	-
98	-1116890434-001	-	-	-
99	-1129556238-001	-	-	-
100	-1142223238-001	-	-	-
101	-1154890434-001	-	-	-
102	-1167556238-001	-	-	-
103	-1180223238-001	-	-	-
104	-1192890434-001	-	-	-
105	-1205556238-001	-	-	-
106	-1218223238-001	-	-	-
107	-1230890434-001	-	-	-
108	-1243556238-001	-	-	-
109	-1256223238-001	-	-	-
110	-1268890434-001	-	-	-
111	-1281556238-001	-	-	-
112	-1294223238-001	-	-	-
113	-1306890434-001	-	-	-
114	-1319556238-001	-	-	-
115	-1332223238-001	-	-	-
116	-1344890434-001	-	-	-
117	-1357556238-001	-	-	-
118	-1370223238-001	-	-	-
119	-1382890434-001	-	-	-
120	-1395556238-001	-	-	-
121	-1408223238-001	-	-	-
122	-1420890434-001	-	-	-
123	-1433556238-001	-	-	-
124	-1446223238-001	-	-	-
125	-1458890434-001	-	-	-
126	-1471556238-001	-	-	-
127	-1484223238-001	-	-	-
128	-1496890434-001	-	-	-
129	-1509556238-001	-	-	-
130	-1522223238-001	-	-	-
131	-1534890434-001	-	-	-
132	-1547556238-001	-	-	-
133	-1560223238-001	-	-	-
134	-1572890434-001	-	-	-
135	-1585556238-001	-	-	-
136	-1598223238-001	-	-	-
137	-1610890434-001	-	-	-
138	-1623556238-001	-	-	-
139	-1636223238-001	-	-	-
140	-1648890434-001	-	-	-
141	-1661556238-001	-	-	-
142	-1674223238-001	-	-	-
143	-1686890434-001	-	-	-
144	-1699556238-001	-	-	-
145	-1712223238-001	-	-	-
146	-1724890434-001	-	-	-
147	-1737556238-001	-	-	-
148	-1750223238-001	-	-	-
149	-1762890434-001	-	-	-
150	-1775556238-001	-	-	-
151	-1788223238-001	-	-	-
152	-1799556238-001	-	-	-
153	-1812223238-001	-	-	-
154	-1824890434-001	-	-	-
155	-1837556238-001	-	-	-
156	-1850223238-001	-	-	-
157	-1862890434-001	-	-	-
158	-1875556238-001	-	-	-
159	-1888223238-001	-	-	-
160	-1899556238-001	-	-	-
161	-1912223238-001	-	-	-
162	-1924890434-001	-	-	-
163	-1937556238-001	-	-	-
164	-1950223238-001	-	-	-
165	-1962890434-001	-	-	-
166	-1975556238-001	-	-	-
167	-1988223238-001	-	-	-
168	-1999556238-001	-	-	-
169	-2012223238-001	-	-	-
170	-2024890434-001	-	-	-
171	-2037556238-001	-	-	-
172	-2050223238-001	-	-	-
173	-2062890434-001	-	-	-
174	-2075556238-001	-	-	-
175	-2088223238-001	-	-	-
176	-2099556238-001	-	-	-
177	-2112223238-001	-	-	-
178	-2124890434-001	-	-	-
179	-2137556238-001	-	-	-
180	-2150223238-001	-	-	-
181	-2162890434-001	-	-	-
182	-2175556238-001	-	-	-
183	-2188223238-001	-	-	-
184	-2199556238-001	-	-	-
185	-2212223238-001	-	-	-
186	-2224890434-001	-	-	-
187	-2237556238-001	-	-	-
188	-2250223238-001	-	-	-
189	-2262890434-001	-	-	-
190	-2275556238-001	-	-	-
191	-2288223238-001	-	-	-
192	-2299556238-001	-	-	-
193	-2312223238-001	-	-	-
194	-2324890434-001	-	-	-
195	-2337556238-001	-	-	-
196	-2350223238-001	-	-	-
197	-2362890434-001	-	-	-
198	-2375556238-001	-	-	-
199	-2388223238-001	-	-	-
200	-2399556238-001	-	-	-
201	-2412223238-001	-	-	-
202	-2424890434-001	-	-	-
203	-2437556238-001	-	-	-
204	-2450223238-001	-	-	-
205	-2462890434-001	-	-	-
206	-2475556238-001	-	-	-
207	-2488223238-001	-	-	-
208	-2499556238-001	-	-	-
209	-2512223238-001	-	-	-
210	-2524890434-001	-	-	-
211	-2537556238-001	-	-	-
212	-2550223238-001	-	-	-
213	-2562890434-001	-	-	-
214	-2575556238-001	-	-	-
215	-2588223238-001	-	-	-
216	-2599556238-001	-	-	-
217	-2612223238-001	-	-	-
218	-2624890434-001	-	-	-
219	-2637556238-001	-	-	-
220	-2650223238-001	-	-	-
221	-2662890434-001	-	-	-
222	-2675556238-001	-	-	-
223	-2688223238-001	-	-	-
224	-2699556238-001	-	-	-
225	-2712223238-001	-	-	-
226	-2724890434-001	-	-	-
227	-2737556238-001	-	-	-
228	-2750223238-001	-	-	-
229	-2762890434-001	-	-	-
230	-2775556238-001	-	-	-
231	-2788223238-001	-	-	-
232	-2799556238-001	-	-	-
233	-2812223238-001	-	-	-
234	-2824890434-001	-	-	-
235	-2837556238-001	-	-	-
236	-2850223238-001	-	-	-
237	-2862890434-001	-	-	-
238	-2875556238-001	-	-	-
239	-2888223238-001	-	-	-
240	-2899556238-001	-	-	-
241	-2912223238-001	-	-	-
242	-2924890434-001	-	-	-
243	-2937556238-001	-	-	-
244	-2950223238-001	-	-	-
245	-2962890434-001	-	-	-
246	-2975556238-001	-	-	-
247	-2988223238-001	-	-	-
248	-2999556238-001	-	-	-
249	-3012223238-001	-	-	-
250	-3024890434-001	-	-	-
251	-3037556238-001	-	-	-
252	-3050223238-001	-	-	-
253	-3062890434-001	-	-	-
254	-3075556238-001	-	-	-
255	-3088223238-001	-	-	-
256	-3099556238-001	-	-	-
257	-3112223238-001	-	-	-
258	-3124890434-001	-	-	-
259	-3137556238-001	-	-	-
260	-3150223238-001	-	-	-
261	-3162890434-001	-	-	-
262	-3175556238-001	-	-	-
263	-3188223238-001	-	-	-
264	-3199556238-001	-	-	-
265	-3212223238-001	-	-	-
266	-3224890434-001	-	-	-
267				

Table 2 (Continued)

r	$a_0(r)$	$b_0(r+1)$	$b_1(r+1)$	$b_2(r+1)$	$b_3(r+1)$
74	-.6937299655-002	-.2683316399+009	.1321463369+011	.7319443399+009	-.5642000000+007
75	-.6342452996-002	-.903590749+009	.1392107963+011	.8132233669+009	-.5792430000+007
76	-.6750162673-002	-.9392632193+009	.1465345117+011	.8453331673+009	-.5944944000+007
77	-.6660126605-002	-.9750630043+009	.1542473485+011	.8732729243+009	-.6099300000+007
78	-.6572846321-002	-.1013401310+010	.1622067676+011	.9120573497+009	-.6255800000+007
79	-.6487632163-002	-.1051290015+010	.1704703244+011	.9466971346+009	-.6414200000+007
80	-.64026604955-002	-.1091341000+010	.1790456697+011	.98220292716+039	-.65724540000+007
81	-.63236669690-002	-.1131266145+010	.1879406494+011	.101354835653+010	-.6736940000+007
82	-.6244752977-002	-.113127329+010	.1971623042+011	.1055355569+010	-.6901256000+007
83	-.6167729232-002	-.1216536434+010	.206720369+011	.1094023714+010	-.7057600000+007
84	-.6092646693-002	-.1269035339+010	.216621274+011	.1133106679+010	-.7235600000+007
85	-.6019346704-002	-.1303445924+010	.2268736526+011	.1173097156+010	-.74061460000+007
86	-.5947434011-002	-.13439239069+010	.2374457166+011	.1214023432+010	-.7578440000+007
87	-.5877961627-002	-.1395439654+010	.2404657353+011	.125589144+010	-.7732640000+007
88	-.5899710205-002	-.1443016539+010	.2598222697+011	.1296710579+010	-.7920940000+007
89	-.57430224079-002	-.1491652664+010	.2715636759+011	.1342492024+010	-.8102106050+007
90	-.5577252107-002	-.1541349449+010	.2836906605+011	.138724844-1+010	-.8262236000+007
91	-.5614441033-002	-.1592209994+010	.2962357533+011	.1432954522+010	-.8469436060+007
92	-.5551633057-002	-.164134979+010	.3091809194+011	.1479216569+010	-.8653561050+007
93	-.5490011705-002	-.1697176634+010	.3223551967+011	.152454444+010	-.8927360000+007
94	-.5431302520-002	-.1751846939+010	.3361430999+011	.1562307669+010	-.9027360000+007
95	-.53729722951-002	-.1806658774+010	.35066637369+011	.1625943326+010	-.9217350000+007
96	-.5315422254-002	-.1856120919+010	.3652667995+011	.1676834102+010	-.9406994659+007
97	-.5256691395-002	-.1920243304+010	.3834042040+011	.17233633704+010	-.9603923565+007
98	-.5205262959-002	-.1979651399+010	.3960033613+011	.1781591609+010	-.9800000000+007
99	-.5156610275-002	-.203264314+010	.4120372294+011	.1835545414+010	-.9990000000+007
100	-.5099151339-002	-.2106734699+010	.42306510353+011	.1906658311+010	-.1019244000+007
101	-.5047206696-002	-.2243137834+010	.44337103952+011	.1946319092+010	-.1030351000+007
102	-.499737465-002	-.2283764629+010	.4632730316+011	.2004033149+010	-.1060351000+007
103	-.4647651182-002	-.2291626934+010	.4810497541+011	.2062459124+010	-.1030930000+007
104	-.4399393632-002	-.23057236639+010	.4999693444+011	.2121957359+010	-.1101276000+007
105	-.4355373535-C32	-.23425105623+010	.5190641792+011	.2132549396+010	-.1123263000+007
106	-.4203636690-002	-.2493245760+010	.5362634304+011	.2244365926+010	-.1143394000+007
107	-.42595666923-002	-.2463366393+010	.5539963647+011	.2307296793+010	-.1165828100+007
108	-.4214633292-002	-.2523437053+010	.5797935339+011	.2321393033+010	-.1165910000+007
109	-.4670666935-002	-.2767241963+010	.6011663229+011	.2366665403+010	-.1264690600+007

Table 2 (Concluded).

r	$a_0(r)$	$a_1(r)$	$b_0(r+2)$	$b_1(r+2)$	$b_2(r+2)$
0	.4940000000+004	-.8330000000+003	-.7092000000+004	-.4933000000+003	-.677717232323+005
1	.1030000000+005	-.3659200000+005	-.677717232323+005	-.9931672545+005	-.1030000000+005
2	-.2112111124002	-.6626556693+005	-.5774950653+006	-.1030530144+006	-.1030000000+005
3	-.3297022352-001	-.362323539972+001	-.623344424974+006	-.3470012936+006	-.2470012936+006
4	-.6058111270+000	-.87333549399+000	-.130655511234+007	-.65245436520+006	-.40532543620+006
5	-.94208910379-001	-.1172116431+004	-.2031229773+007	-.94897293099+006	-.4544070000100
6	-.112333604+000	-.2564964242+000	-.3169356704+007	-.129643210+007	-.12359290000000+005
7	-.17251531011-001	-.7295094422+000	-.4701912519+007	-.1694029500+007	-.351693000000+005
8	-.53110595990-001	-.6351191009+000	-.67297029624+007	-.2226039909+007	-.1679600000+006
9	-.407613532-001	-.6169136336+000	-.9350461792+007	-.32769212640+002	-.1296300000+006
10	-.3263552931-001	-.5596771105+000	-.32026212336+002	-.1503453660+006	-.1503453660+006
11	-.2682524642-001	-.6121465455+000	-.13676353474+008	-.44672632553+002	-.1294613000+006
12	-.2212724939-001	-.4712226161+000	-.163252126034+008	-.50321214851+002	-.3069613000+006
13	-.18664673361-001	-.4372036440+000	-.21292234990+008	-.7152513000+002	-.2266000030+008
14	-.1594611357-091	-.4071469138+000	-.2811400024+008	-.4826936929+002	-.2062330000+006
15	-.1377134995-001	-.3301456082+000	-.355323650039+008	-.16232206534+008	-.3010400000+006
16	-.1280754055-001	-.437163924+000	-.44355399303+008	-.12032923535+003	-.3976500000+006
17	-.1065750761-001	-.32269631175+000	-.64732138974+003	-.1407631113+003	-.3752030000+006
18	-.9351610588-002	-.3164939239+000	-.6624232072+002	-.1631261462+003	-.4443900000+007
19	-.3928422229-002	-.3013374312+000	-.8932345630+003	-.18726937532+003	-.456339350000+006
20	-.7479231671-002	-.2863217924+000	-.97012389018+003	-.21472360320+003	-.409934000000+006
21	-.6744053263-002	-.27202362772+000	-.111452238927+009	-.24440301432+003	-.4446790600+005
22	-.6112224974-002	-.26640303474+000	-.1664797433+009	-.2761614432+003	-.3946530000+006
23	-.566369169-002	-.3494236474+003	-.1602322115+009	-.3103550060+003	-.6625600000+006
24	-.5034057140-002	-.3369703485+000	-.16392348705+009	-.3443103294+003	-.6939300000+006
25	-.4684315419-002	-.2293670622+000	-.31694366562+000	-.3642619212+003	-.7474540000+006
26	-.4294059223-002	-.2204336392+000	-.2360438144+009	-.4319372901+003	-.8023613600+006
27	-.3966422663-002	-.2121733352+000	-.2376043835+009	-.4732323459+003	-.8603600030+006
28	-.3672352412-002	-.2045030212+000	-.32823020223+009	-.5330673994+003	-.9197030000+006
29	-.3411213639-002	-.1923614369+000	-.37433171364+039	-.5310257623+003	-.9310376000+006
30	-.3170825239-002	-.1906239138+000	-.4244236950+009	-.6374037474+003	-.1044360039+007
31	-.2964696210-002	-.1844429202+000	-.4294105475+009	-.6973141629+003	-.1109263600+006
32	-.276375160-002	-.1783623393+000	-.53956261463+009	-.76036729329+003	-.1177165000+006
33	-.2592159362-002	-.1700329413+000	-.6052733036+009	-.8650161294+009	-.1246640000+100
34	-.2543019699-002	-.1672035460+000	-.7445675454+009	-.9392291635+003	-.1317679030+007
35	-.2295034093-002	-.16302420400+000	-.8332361923+639	-.9743764935+003	-.1390940000+007

Table 3. Numerical coefficients of the (0,2)-factorization of the JYIK operator with $x_1 = 1$ and $x_2 = 10$.

r	$a_0(r)$	$a_1(r)$	$b_0(r+2)$	$b_1(r+2)$	$b_2(r+2)$
37	.2645573795-002	-.1540531999+000	.9298463247+009	.1053524312+009	-.1466130000+007
38	.1934577302-002	-.1499230312+000	.162203054+010	.16343400000+007	-.16343400000+007
39	.1632324283-002	-.1460056740+000	.1134161703+010	.1224450148+009	-.162260000+007
40	.1237925201-002	-.1422352217+000	.1243167432+010	.1316434999+009	-.17047303600+007
41	.1650599595-002	-.1367474906+000	.1370554416+010	.1412910523+009	-.1736940000+007
42	.1669661203-002	-.13507293425+000	.1501746644+010	.16139363652+009	-.1872636000+007
43	.1494504689-002	-.1321690319+000	.1642157546+010	.1619761290+009	-.1959200000+007
44	.1424594494-002	-.1291053229+000	.1792210043+010	.1730350359+009	-.2044630000+007
45	.1359465634-002	-.1261729233+000	.1932336532+010	.1845857777+009	-.2139340000+007
46	.1293664742-002	-.1230323863+000	.2132979088+010	.1966390462+009	-.2232440000+007
47	.12416245295-002	-.1207030172+000	.2304538296+010	.2092056333+009	-.2332724356000+007
48	.1133559349-002	-.1181403007+000	.2497627253+010	.2322959263+009	-.2424600000+007
49	.1138406097-002	-.1165013306+000	.2702564320+010	.2353500090+009	-.2523500090+007
50	.1092012425-002	-.1133217499+000	.2919389132+010	.2500912246+009	-.2624436000+007
51	.1044934261-002	-.1110556977+000	.3150064152+010	.26403175045+009	-.2722446000+007
52	.1006650913-002	-.1038777618+000	.3393615339+010	.2801104624+009	-.2832383030+007
53	.9676628325-003	-.1067829373+000	.3651042157+010	.2959897901+009	-.2939300000+007
54	.9303923309-003	-.1042665346+000	.3922862574+010	.3124391796+009	-.30448206000+007
55	.8961660422-003	-.1026244167+000	.4207604062+010	.3294963226+009	-.3159032000+007
56	.8633440341-003	-.1009534233+000	.45114004593+010	.3471659112+009	-.32271940000+007
57	.8325273002-002	-.9914691230-001	.48310097660+010	.3654496373+009	-.33067303000+007
58	.8923718225-003	-.9740441019-001	.5164272233+010	.3843671922+009	-.3653660000+007
59	.7749042282-003	-.9572169939-001	.5516662396+010	.4039262395+009	-.3622460000+007
60	.7495209077-003	-.9409577269-001	.532762543369+010	.4241375595+009	-.3743139000+007
61	.7233266090-003	-.9252332121-001	.6274131420+010	.44501112547+009	-.39659466469+007
62	.699440655-003	-.9100321359-001	.66310267953+010	.4665595470+009	-.39906436640+007
63	.6767629996-003	-.8953143911-001	.710712244374+010	.4837916235+009	-.4417420000+007
64	.6551196032-003	-.8810634020-001	.7593463616+010	.51112136910+009	-.42461600001007
65	.6344954425-003	-.8672560153-001	.8020617056+010	.5853514264+009	-.43265300000+007
66	.61426279260-003	-.85366233952-001	.85043921623+010	.6397003260+009	-.4509440000+007
67	.59200522235-003	-.8403934206-001	.9019318232+010	.6847766643+009	-.4643086000+007
68	.5724433651-003	-.8263011004-001	.9562467924+010	.6105906903+009	-.4730700000+007
69	.5510047026-003	-.8160734394-001	.101070424+011	.6371529374+009	-.4919300004009
70	.54462306459-003	-.8042696143-001	.1063901313+011	.66447244172+009	-.5059340000+007
71	.52392426433-003	-.7926794050-001	.11323902121+011	.6925657217+009	-.5202440000+007
72	.51393866924-003	-.7814236323-001	.1192400651+011	.7214375420+009	-.5346240000+007
73	.49959733084-003	-.7701733496-001	.1253065355+011	.7511003730+009	-.5493560000+007

Table 3 (Continued)

r	$a_0(r)$	$b_0(r+2)$	$b_1(r+2)$	$b_2(r+2)$
72	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
73	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
74	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
75	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
76	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
77	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
78	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
79	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
80	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
81	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
82	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
83	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
84	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
85	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
86	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
87	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
88	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
89	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
90	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
91	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
92	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
93	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
94	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
95	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
96	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
97	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
98	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
99	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
100	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
101	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004
102	-0.9357743544	-0.5938243401	-0.2516329315	-0.0645300004

Table 3 (Concluded).

r	$a_0(r)$	$a_1(r)$	$a_2(r)$	$b_0(r+3)$	$b_1(r+3)$
0	$-4940000000+004$	$-8562000000+004$	$-2630000000+003$	$-7092000000+004$	$-9800930000+003$
1	$-1032900000+005$	$-4562290000+005$	$-2658206430+001$	$.3997304230+005$	$-4940000000+004$
2	$-1830000000+005$	$-1712536120+002$	$.1594028034+001$	$.1075944935+006$	$-1030000000+005$
3	$-404631331+001$	$-61640097260+001$	$.1738469966+001$	$.2237645532+006$	$-1830000000+005$
4	$-1015184176+001$	$-4031667709+001$	$.1912950317+001$	$.3993334754+006$	$-2370000000+005$
5	$.5059738235+000$	$-3177135479+001$	$.2096037422+001$	$.6463656740+006$	$-40548045600+005$
6	$.3140941673+000$	$-2639663183+001$	$.2236196675+001$	$.9765013367+006$	$-54444600306+005$
7	$.2205399979+000$	$-2383691453+001$	$.24763910314+001$	$.1401631943+002$	$-7023000000+005$
8	$.1669733692+000$	$-217863930+001$	$.26693600309+001$	$.19337323677+007$	$-810000000+005$
9	$.13200056954+000$	$-20236653833+001$	$.26331583373+001$	$.2604631253+007$	$-107900000+006$
10	$.1094684553+000$	$-1915033116+001$	$.3057964639+001$	$.3366266390+007$	$-1294820000+006$
11	$.9265772490+001$	$-182637504+001$	$.3256534679+001$	$.42992033971+007$	$-4534490000+006$
12	$.80043346332+001$	$-1754615377+001$	$.3449764714+001$	$.5363994393+007$	$-1791440000+006$
13	$.7028558242+001$	$-1695067554+001$	$.36464727339+001$	$.6613895102+007$	$-2069000000+006$
14	$.6253016946+001$	$-1646760796+001$	$.38435383340+001$	$.39369923531+007$	$-2366000000+006$
15	$.5623774551+001$	$-1605107302+001$	$.4041040343+001$	$.9650132145+007$	$-2632230300+006$
16	$.51041027223+001$	$-1569334572+001$	$.423872602+001$	$.1146532591+006$	$-3019400000+006$
17	$.4668409464+001$	$-153279203+001$	$.4336741776+001$	$.1349430931+003$	$-3375440000+006$
18	$.4298364464+001$	$-1511069353+001$	$.46349112674+001$	$.1574901328+003$	$-3752600000+006$
19	$.3920332567+001$	$-148703425+001$	$.4833256726+001$	$.1824131792+008$	$-4143000000+006$
20	$.3704320824+001$	$-1463643627+001$	$.503175052+001$	$.2090310211+603$	$-4563230000+006$
21	$.3463531950+001$	$-1445493370+001$	$.5229374669+001$	$.2390624636+003$	$-4999400000+006$
22	$.3250361602+001$	$-1429230793+001$	$.5429112637+001$	$.2726263063+003$	$-5434400000+006$
23	$.30619327673+001$	$-1413636181+001$	$.5627951628+001$	$.30022413505+003$	$-5930000000+006$
24	$.2893241336+001$	$-1399433335+001$	$.58326379334+601$	$.3466263947+602$	$-64250063000+006$
25	$.2741623307+001$	$-1386439433+001$	$.6025036235+001$	$.3035002394+602$	$-6939200000+006$
26	$.2604712140+001$	$-1374561643+001$	$.6224963299+001$	$.4323016044+603$	$-7424400000+006$
27	$.2430491279+001$	$-136361433+001$	$.6424404914+001$	$.4015695293+603$	$-8023000000+006$
28	$.236730337+001$	$-1353500420+001$	$.6623036394+601$	$.5332426754+602$	$-8603600000+006$
29	$.2205723173+001$	$-1344135522+001$	$.6822552760+601$	$.5334596214+603$	$-91976363000+006$
30	$.2163734252+001$	$-1335492424+001$	$.70218349256+601$	$.6473594676+603$	$-9310330000+006$
31	$.20319191659+001$	$-13273261739+001$	$.7221434236+601$	$.7100609140+603$	$-1044428000+007$
32	$.1900307170+001$	$-1319774636+001$	$.7420566110+001$	$.7766421607+603$	$-1109213000+007$
33	$.1826389962+001$	$-1312629072+001$	$.7619979412+001$	$.84731340754+603$	$-1477166000+007$
34	$.1656723113+001$	$-1306064499+001$	$.7819425249+001$	$.9221623545+603$	$-1246400000+007$
35	$.12903247741+001$	$-1299320934+001$	$.8016900271+001$	$.1001253702+009$	$-1347630000+007$
36	$.1230324793+001$	$-1293263249+001$	$.8216404243+001$	$.1024750149+009$	$-1390920000+007$

Table 4. Numerical coefficients of the $(0, 3)$ -factorization of the JYIK operator with $x_1 = 1$ and $x_2 = 10$.

r	$a_0(r)$	$a_1(r)$	$a_2(r)$	$b_0(r+3)$	$b_1(r+3)$
37	.1673674e07-001	-.1232430044e001	.8417932944e001	-.1172755996e009	-.1466120000e007
38	.1620560198e-001	-.123203568e+001	.3617435170e+001	.1265395044e009	-.1543450000e007
39	.1570665826e-001	-.1273259019e+001	.8817059205e+001	.1362766092e+009	-.1622600000e007
40	.1523709966e-001	-.1273574197e+001	.9016663494e+001	.1465047940e+009	-.1703734060e+009
41	.1479443126e-001	-.1269129171e+001	.9276266625e+001	.1572993018e+009	-.1735940000e007
42	.1431734084e-001	-.1264906999e+001	.9415397312e+001	.1624659236e+009	-.1822033503e+007
43	.1393111246e-001	-.12606313635e+001	.9615544404e+001	.1802246204e+009	-.1959239039e+009
44	.1360666961e-001	-.1257061969e+001	.98163068894e+001	.19251729332e+009	-.20443707090e+009
45	.1325153200e-001	-.1253431355e+001	.100140034556e+002	.2053372120e+009	-.2139340000e+007
46	.1291432320e-001	-.1249929948e+001	.1021457327e+002	.2187353623e+009	-.2232446900e+007
47	.1259363193e-001	-.1246601330e+001	.10414227661e+002	.2327242477e+009	-.2327469000e+007
48	.1229231241e-001	-.1243412232e+001	.1061399132e+002	.2472242525e+009	-.2424530000e+007
49	.1199730131e-001	-.1240363460e+001	.10813717214e+002	.2624122574e+009	-.2523530000e+007
50	.1171962108e-001	-.1237446556e+001	.1101845363e+002	.2701696422e+009	-.2624445900e+007
51	.1145429413e-001	-.1234642768e+001	.1121319994e+002	.2945372321e+009	-.2722743060e+007
52	.11200772859e-001	-.1231952963e+001	.1141295572e+002	.3115653720e+009	-.2832384000e+007
53	.10953396138e-001	-.1229326756e+001	.11612722034e+002	.32911243726e+009	-.2939360000e+007
54	.1072555031e-001	-.1226221490e+001	.1181249336e+002	.34742675617e+009	-.3044259000e+007
55	.10502620637e-001	-.1224439136e+001	.12012274334e+002	.3664253566e+009	-.3159020000e+007
56	.1029396529e-001	-.1222183300e+001	.1221206234e+002	.3866607115e+009	-.3221910000e+007
57	.1003124282e-001	-.1219965158e+001	.12411050524e+002	.4063045264e+009	-.3363213600e+007
58	.9826773074e-002	-.1217824234e+001	.1261166101e+002	.4274103013e+009	-.3503660000e+007
59	.9695322617e-002	-.1215753362e+001	.1281146696e+002	.4461479062e+009	-.3632243006e+007
60	.9513001428e-002	-.1213763666e+001	.1301126034e+002	.4716104911e+009	-.3744312900e+007
61	.9336022610e-002	-.1211626520e+001	.1321110666e+002	.4946395360e+009	-.3865942000e+007
62	.9167156124e-002	-.1209973577e+001	.1341023265e+002	.5132572309e+009	-.3996632600e+007
63	.9003194463e-002	-.1208121659e+001	.1361026453e+002	.5434613253e+009	-.4112460000e+007
64	.8845321500e-002	-.1206427322e+001	.1381060152e+002	.56339452302e+009	-.4246140000e+007
65	.8692272532e-002	-.1204739305e+001	.1401042432e+002	.5952094156e+009	-.4376232000e+007
66	.8543521510e-002	-.1203103616e+001	.1421029303e+002	.6222695605e+009	-.4509420000e+007
67	.8402142025e-002	-.1201513024e+001	.14410149324e+002	.6501374545e+009	-.4644630000e+007
68	.8264022214e-002	-.11999202545e+001	.14609996603e+002	.6732362504e+009	-.4782230364e+007
69	.8124021372e-002	-.11924232932e+001	.14899338531e+002	.7033246553e+009	-.4919300000e+007
70	.8006666640e-002	-.1197041161e+001	.1500971050e+002	.73627234922e+009	-.5059236050e+007
71	.7876011265e-002	-.1195635329e+001	.1520960344e+002	.7699254652e+009	-.5205360000e+007
72	.7755400545e-002	-.1194269639e+001	.1540945590e+002	.8020103701e+009	-.5346660000e+007
73	.7637666006e-002	-.1192942393e+001	.15609632994e+002	.8349753750e+009	-.5493400000e+007

Table 4 (Continued)

r	$a_0(r)$	$a_1(r)$	$a_2(r)$	$b_0(r+3)$	$b_1(r+3)$
74	-7528429960-002	-1191652003+001	-1530920724+002	.856380000000+009	-.564200000000+007
75	-7412642924-002	-1190396942+001	-1600903721+002	.9035092649+009	-.5792400000+007
76	-73040362134-002	-1169125734+001	-1620897126+002	.9392624093+009	-.5944940000+007
77	-7200250526-002	-1157987173+001	-1640855775+002	.9753621943+009	-.6099300000+007
78	-7098579243-002	-1136329325+001	-1660874707+002	.1013400500+010	-.6256050000+007
79	-6995126524-002	-1135702624+001	-1630633913+002	.1051009205+010	-.6414200000+007
80	-6903672292-002	-1134604113+001	-1700853381+002	.1091240190+010	-.6574500000+007
81	-6316022293-002	-1133533493+001	-1720643104+002	.11312765335+010	-.6736941000+007
82	-67139503523-002	-1132459636+001	-1740033071+002	.11731210000+007	-.6901210000+007
83	-6630238217-692	-1131471536+001	-1760823274+002	.1215525624+010	-.7067600000+007
84	-6549318403-902	-1130472258+001	-1780813704+002	.1259004529+010	-.7235960000+007
85	-6459761063-002	-1129503904+001	-1810804356+002	.1303445144+010	-.7406100000+007
86	-6327732547-002	-11285622620+001	-1820795218+002	.1343919259+010	-.7528450000+007
87	-6297763356-002	-11276343593+001	-18407926206+002	.1395436324+010	-.7752640000+007
88	-6219747290-002	-1126736046+001	-18607273534+002	.1443015749+010	-.7928990000+007
89	-6143644312-002	-1125354239+001	-1889769911+002	.1491661354+010	-.8107100000+007
90	-6069373329-002	-11249922466+001	-1909266355+002	.1541339039+010	-.8292220000+007
91	-5996371486-002	-1124150052+001	-1920752479+002	.1592269134+010	-.8469449000+007
92	-5926074935-002	-1123263854+001	-1940744377+002	.1644134169+010	-.8653563200+007
93	-5856925001-002	-112326747+001	-1960736643+002	.16921745374+010	-.8839700000+007
94	-5789364923-002	-1121232650+001	-19807238972+002	.1751346129+010	-.9023030000+007
95	-5723340692-002	-1120961496+001	-20007214359+002	.1806656964+010	-.9217332000+007
96	-5652300686-002	-112029674+001	-2020714100+002	.1863120109+010	-.9409940000+007
97	-5506695539-002	-1169462327+001	-2040706339+002	.1920747494+010	-.9603983000+007
98	-5530922060-002	-1163244397+001	-2060699822+002	.197950994+010	-.9833010000+007
99	-5473603093-002	-1162303630+001	-2080692295+002	.2029542504+010	-.9930000000+007
100	-5414622436-002	-1167341720+001	-2100646104+002	.2160733039+010	-.1019720000+007
101	-5356709681-002	-1166661627+001	-2120679445+002	.2163157034+010	-.1039994000+007
102	-53001010137-002	-116595133+001	-2140672914+002	.222676319+010	-.1060640050+007
103	-5244690943-002	-1163941633+001	-216066507+002	.2291626124+010	-.1080930050+007
104	-5190415806-002	-1164201323+001	-2180660321+002	.2357235329+010	-.1161270000+007
105	-5137249910-002	-1164673273+001	-2200654652+002	.23436104314+010	-.1123750000+007
106	-5035152601-002	-1163457292+001	-2220647793+002	.2493744953+010	-.1143944000+007
107	-5034169359-002	-1162303636+001	-2240642055+002	.2563666143+010	-.1165824000+007
108	-49240723233-002	-1162260179+001	-2260636219+002	.2634836243+010	-.1186940000+007

Table 4 (Concluded).

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$u_{r,1}(100)$	$J_r(1)$
0	(- .7651976936+000)				
1	- .3730376572+000				
2	- .2437001044+004				
3	- .1503541524+004				
4	- .1011048400+004				
5	- .2324395430+002				
6	- .1207342004+002				
7	- .1411200440+002				
8	- .1181962353+000				
9	- .6038466394+000				
10	- .6038466394+000				
20	- .6351821214+017				
30	- .1625626202+032				
40	- .1353266430+003				
50	- .1331955091+002				
60	- .4195295522+002				
70	- .4491654953+112				
80	- .1997083693+133				
90	- .232322424+156				
91	- .4125623630+153				
92	- .232322424+160				
93	- .1532103133+160				
94	- .1532103133+162				
95	- .3034103043+165				
96	- .3034103043+167				
97	- .1093235416+167				
98	- .1093235416+225				
99	- .3002416235+162				
100	- .1527633235+123				
101	- .7621720601+181				
102	- .3963504232+183				
103	- .1920636955+183				
104	- .92952731+183				
105	- .4335670105+190				
106	- .2321854105+192				
107	- .1123033691+194				
108	- .5354635093+197				
109	- .2523661232+199				
110	- .1129102009+201				
111	- .1339023360+012				
112	- .1339023360+012				

Table 5. Computation of $J_r(1)$ from the JYIK equation with $x_1=1$, $x_2=10$, $i=0$, $m=100$, $j=1$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$\eta_{r,1}(100)$	$I_r(10)$
0	(.2815716623+004)				
1	(.2670983304+004)				
2	.92324056477+002				
3	-.5042339146+009				
4	.4524550139+006				
5	.6103432477+009				
6	3593709842+009				
7	.37306007021+009				
8	.27309040324+009				
9	.1926223969+009				
10	.11216238321+009				
29	.67567025607+004				
39	.2619197236-002				
40	.2127585141-010				
50	.1103855353-019				
60	.8146012303-030				
70	.9640398631-041				
80	.2568879314-052				
90	.10640340501-060				
91	.1069923979-065				
92	.6056290601-060				
93	.3389912566-069				
94	.1326514763-069				
95	.102242424039-070				
96	.5564569099-072				
97	.2281621020-073				
98	.16309727439-074				
99	.5522632321-075				
100	.43045402457-076				
101	.2312373375-070	-.4312543353-001	.1033742137-037	-.2493840369-090	.5345192136-009
102	.1125329831-079	.2070337364-002	.1082336346-087	.5669009436-093	.2613926261-090
103	.3666466404-081	-.9726392768-004	.1022344216-037	-.1368599159-095	.1264624676-091
104	.2324923555-082	.4532505249-005	.1032344202-037	.3225957327-093	.9000000000
105			.1032344202-037	.0000000000	

Table 6. Computation of $I_r(10)$ from the JYIK equation with $x_1=1$, $x_2=10$, $i=0$, $m=100$, $j=2$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$n_{r,1}(100)$	$(-)^{r_K}_r(10)$
0					
1	(-.1726006232-004)				
2	(-.1254857345-006)				
3	(-.21502031201-006)				
4	(-.2305035564-006)				
5	(-.13976925663001)				
6	(-.72291742805-001)				
7	(-.3458100654-000)				
8	(-.2232495045+002)				
9	(-.2461457645+003)				
10	(-.7912490584-003)				
20	(-.25002610234019)				
30	(-.10782723164018)				
40	(-.76715092464-026)				
50	(-.52519420164-036)				
60	(-.22444735514+037)				
70	(-.5055052464-058)				
80	(-.4232361924-070)				
90	(-.11631095564+083)				
91	(-.321294492274083)				
92	(-.4175273424-083)				
93	(-.79635446114-083)				
94	(-.16345043524-083)				
95	(-.29824321634-089)				
96	(-.50751546624-090)				
97	(-.14672395434-092)				
98	(-.20429030064-092)				
99	(-.62668282364-094)				
100	(-.97142014464-096)				
101	(-.30075454974-092)				
102	(-.41322374664-093)				
103	(-.89241129421+092)				
104	(-.182357201064-101)				
105	(-.402522230084-102)				
106	(-.82181664924-103)				
107	(-.10604931034+103)				
108	(-.25040934774-103)				
109	(-.9359241054-102)				
110	(-.219619009374-102)				
111	(-.47626923464-110)				
112	(-.20926300014-025)				

Table 7. Computation of $(-)^{r_K}_r(10)$ from the JYIK equation with $x_1=1$, $x_2=10$, $i=0$, $m=100$, $j=3$.

r	z(r)	Y _r (1)
0	(- .8325696420-001)	(- .8325696420-001)
1	(- .7912123213+000)	(- .7912123213+000)
2	(- .6500926010+001)	(- .6500926010+001)
3	(- .5321515064+001)	(- .5321515064+001)
4	(- .3613254574+005)	(- .3613254574+005)
5	(- .1234264934+002)	(- .1234264934+002)
6	(- .000+306000272655)	(- .000+306000272655)
7	(- .423165155+0.1)	(- .423165155+0.1)
8	(- .2261407164+0.12)	(- .2261407164+0.12)
9	(- .0920004692+0.13)	(- .0920004692+0.13)
10	(- .04113820315+0.14)	(- .04113820315+0.14)
20	(- .1248172197+0.29)	(- .1248172197+0.29)
30	(- .28700293373+0.46)	(- .28700293373+0.46)
40	(- .69933729250+0.62)	(- .69933729250+0.62)
50	(- .16911594551+0.74)	(- .16911594551+0.74)
60	(- .4162813390+1.05)	(- .4162813390+1.05)
70	(- .9910402166+1.26)	(- .9910402166+1.26)
80	(- 2.053006531+1.43)	(- 2.053006531+1.43)
90	(- 4.0435954335+1.70)	(- 4.0435954335+1.70)
61	(- 6.03743251+1.91)	(- 6.03743251+1.91)
62	(- 1.0340434351+2.15)	(- 1.0340434351+2.15)
63	(- 2.77143210192+2.31)	(- 2.77143210192+2.31)
64	(- 4.6643546391+2.52)	(- 4.6643546391+2.52)
65	(- 7.2606301620+2.74)	(- 7.2606301620+2.74)
66	(- 11.4233011622+2.95)	(- 11.4233011622+2.95)
67	(- 16.6663011644+3.13)	(- 16.6663011644+3.13)
68	(- 22.9993011666+3.31)	(- 22.9993011666+3.31)
69	(- 30.3323011688+3.49)	(- 30.3323011688+3.49)
70	(- 40.6653011710+3.67)	(- 40.6653011710+3.67)
71	(- 52.9983011732+3.85)	(- 52.9983011732+3.85)
72	(- 66.3313011754+4.03)	(- 66.3313011754+4.03)
73	(- 81.6643011776+4.21)	(- 81.6643011776+4.21)
74	(- 98.9973011798+4.39)	(- 98.9973011798+4.39)
75	(- 117.3303011820+4.57)	(- 117.3303011820+4.57)
76	(- 137.6633011842+4.75)	(- 137.6633011842+4.75)
77	(- 159.9963011864+4.93)	(- 159.9963011864+4.93)
78	(- 184.3293011886+5.11)	(- 184.3293011886+5.11)
79	(- 210.6623011908+5.29)	(- 210.6623011908+5.29)
80	(- 239.9953011930+5.47)	(- 239.9953011930+5.47)
81	(- 271.3283011952+5.65)	(- 271.3283011952+5.65)
82	(- 304.6613011974+5.83)	(- 304.6613011974+5.83)
83	(- 340.9943011996+6.01)	(- 340.9943011996+6.01)
84	(- 380.3273012018+6.19)	(- 380.3273012018+6.19)
85	(- 421.6603012040+6.37)	(- 421.6603012040+6.37)
86	(- 464.9933012062+6.55)	(- 464.9933012062+6.55)
87	(- 510.3263012084+6.73)	(- 510.3263012084+6.73)
88	(- 557.6593012106+6.91)	(- 557.6593012106+6.91)
89	(- 607.9923012128+7.09)	(- 607.9923012128+7.09)
90	(- 660.3253012150+7.27)	(- 660.3253012150+7.27)
91	(- 714.6583012172+7.45)	(- 714.6583012172+7.45)
92	(- 770.9913012194+7.63)	(- 770.9913012194+7.63)
93	(- 829.3243012216+7.81)	(- 829.3243012216+7.81)
94	(- 889.6573012238+7.99)	(- 889.6573012238+7.99)
95	(- 951.9903012260+8.17)	(- 951.9903012260+8.17)
96	(- 1.015.3233012282+8.35)	(- 1.015.3233012282+8.35)
97	(- 1.080.6563012304+8.53)	(- 1.080.6563012304+8.53)
98	(- 1.147.9893012326+8.71)	(- 1.147.9893012326+8.71)
99	(- 1.217.3223012348+8.89)	(- 1.217.3223012348+8.89)
100	(- 1.287.6553012370+9.07)	(- 1.287.6553012370+9.07)

Table 8. Computation of $Y_r(1)$ from the JYIK equation with $x_1=1$, $x_2=10$, $i=0$, $m=100$, $j=4$.

Example 4.1.2 In this example we compute the solutions $I_r(1)$, $J_r(10)$, $Y_r(10)$ and $(-)^r K_r(1)$ of the homogeneous equation defined in Figure 6, with $x_1 = 10$, $x_2 = 1$, for $r = 0, 1, 2, \dots, 100$. Both $I_r(1)$ and $K_r(1)$ are monotonic over this range of r . The solutions $J_r(10)$ and $Y_r(10)$ are monotonic only for $10 \leq r \leq 100$; for lower values they oscillate with approximately the same amplitude. However, since there is no "crossing over" of growth behavior in the four solutions in the nonasymptotic region $0 \leq r \leq 10$, we expect the algorithm to be stable with the following numbers of initial conditions: $j = 1$ for $I_r(1)$, $j = 2$ for $J_r(10)$, $j = 3$ for $Y_r(10)$ and $j = 4$ for $(-)^r K_r(1)$; compare (4.1.4). Tables 9 through 12 give the results of the computation of each of these solutions for selected values of r : see §4.0 for a full description of these tables. Again, the stability of the algorithm for each solution is confirmed by comparison with Tables 9.4 and 9.11 of [16].

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$\eta_{r,1}(100)$	$I_r(1)$
0	0	.1266065373+091)	.5651301041+030	.2216022493-001	.43574266934000
1	.8736497193+004	.6473206716+004	.2737120222-002	.2714631560-003	.2240366143-004
2	.26314315253+004	.6293320543+003	.1599248232-005	.9960323035-007	.6510305364-009
3	.1063373074+003	.1397344051+002	.4273660501+001	.14736652000	.22529932540-009
4	.1397344051+002	.1306266563+000	.991236944-002	.9960323035-007	.6510305364-009
5	.1306266563+000	.991236944-002	.22529932540-009	.2216022493-001	.43574266934000
6	.1306266563+000	.991236944-002	.22529932540-009	.2216022493-001	.43574266934000
7	.1306266563+000	.991236944-002	.22529932540-009	.2216022493-001	.43574266934000
8	.1306266563+000	.991236944-002	.22529932540-009	.2216022493-001	.43574266934000
9	.1306266563+000	.991236944-002	.22529932540-009	.2216022493-001	.43574266934000
10	.1306266563+000	.991236944-002	.22529932540-009	.2216022493-001	.43574266934000
20	.6550496636-012	.7322042364-156	.5452992350-165	.3966335937-024	.3539500539-041
30	.1800612914-023	.4157059733-159	.2993527732-162	.4121360742-059	.1121360742-059
40	.13233724199-160	.22333724199-160	.1627956634-169	.2934633509-029	.2934633509-029
50	.1295333695-162	.1295333695-162	.37522043716-172	.1046639065-099	.1046639065-099
60	.71133012148-165	.4234452563-091	.7096176949-124	.4159346657-142	.4159346657-142
70	.4526633073-112	.4526633073-112	.455036933-176	.1236634041-172	.1236634041-172
80	.1096063917-133	.1096063917-133	.65274242-122-131	.3355733701-133	.3355733701-133
90	.7322042364-156	.7322042364-156	.1694276750-135	.8473624609-132	.8473624609-132
91	.4157059733-159	.4157059733-159	.2993527732-162	.4121360742-059	.1121360742-059
92	.22333724199-160	.22333724199-160	.1627956634-169	.2934633509-029	.2934633509-029
93	.1295333695-162	.1295333695-162	.37522043716-172	.1046639065-099	.1046639065-099
94	.71133012148-165	.4234452563-091	.7096176949-124	.4159346657-142	.4159346657-142
95	.3663451024-167	.3663451024-167	.455036933-176	.1236634041-172	.1236634041-172
96	.2075333425-169	.2075333425-169	.65274242-122-131	.3355733701-133	.3355733701-133
97	.1103261935-171	.1103261935-171	.1694276750-135	.8473624609-132	.8473624609-132
98	.5002305467-174	.5002305467-174	.7624674109-189	.41942766327-199	.2656241350-192
99	.3020337907-176	.3020337907-176	.7624674109-189	.7624674109-189	.99314141429-195
100	.155533273330-173	.155533273330-173	.7624674109-189	.7624674109-189	.4793644220-197
101	.7933236136-181	.1966693337+002	.8473624609-132	.8473624609-132	.22363995539-199
102	.4004297266-163	.3675031733+003	.8465245443-133	.8465245443-133	.107737964-201
103	.2000932222-145	.7236654732+004	.847233134-133	.847233134-133	.4031329633-204
104	.9399233644-163	.1539914453+006	.8473590116-133	.8473590116-133	.2304364468-206
105	.4350352452-190	.3766633026+008	.7562309127-193	.7562309127-193	.9626193346-209
106	.2353066408-192	.6532297221+008	.7435165105-194	.7435165105-194	.60600000000
107	.1430655366-194	.1366545395+010	.7513330192-195	.7513330192-195	.60000000000
108	.5301449102-192	.2344063443+011	.8473673392-133	.8473673392-133	.60000000000
109	.2336934367-199	.6019635172+012	.8473673400-143	.8473673400-143	.60000000000
110	.1103352947-201	.4206171766+013	.8473673400-143	.8473673400-143	.60000000000
111		.4423624610-183			
112					

Table 9. Computation of $I_r(1)$ from the JYIK equation with $x_1=10$, $x_2=1$, $i=0$, $m=100$, $j=1$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$n_{r,1}(100)$	$J_r(10)$
0	(-.24593857645+000)				
1	(-.43245374612-001)				
2	-.2436172043e+004		.26465038132+000	.6600	
3	-.1321122902e+005		.1600-1.1931-000	.6600	
4	-.393030111054005		-.2196045600		
5	.339520092284005		-.33405153133+000		
6	.105217447204006		-.14466942309-000		
7	-.32200022114e+006		.21621631622-000		
8	-.1223209429e+006		.34235412694000		
9	-.1517728304e+006		.291635535324000		
10	-.1690396443e+006		.267420010324000		
20	-.63619226652+000		.1151336725-004		
30	-.6700566553-003		.1651096265-011		
40	-.59301240473-011		.6030693313-020		
50	-.43320222CD1-020		.1284613508-030		
60	-.3499165455-030		.6909439250-040		
70	-.4627963205-041		.4963191495-051		
80	-.13353C9550-052		.34339569422-063		
90	-.1060675200-064		.4129211626-075		
91	-.61454327212-065		.8221660249-076		
92	-.3299217210-066		.12409060529-077		
93	-.1930326640-066		.663505265036-060		
94	-.109706313-069		.3667462936-060		
95	-.6940264240-071		.4242220224-061		
96	-.3250100141-072		.93329260126-062		
97	-.177238112-073		.3031274512-054		
98	-.9342805005-075		.2599477201-066		
99	-.29683542887-076		.19164192351-066		
100	-.26600166963-076		.6597316665-QC9		
101	-.1342369729-076	.4336281742-001	.1474190197-090		
102	-.6036390120-076	.2091657372-002	.6597279630-003	.3228931974-009	
103	-.3423219014-076	.9570563604-004	.6597315928-003	.4603712204-000	
104	-.1739735391-076	.46225C7666-005	.6597316065-003	.7311103414-002	
105				.0060606060	

Table 10. Computation of $J_r(10)$ from the JYIK equation with $x_1=10$, $x_2=1$, $i=0$, $m=100$, $j=2$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$n_{r,1}(100)$	$Y_r(10)$
0	(.5567116230+001)				
1	(-.34901542442+000)				
2	(-.5866002460-002)				
3	- .40244246442+004				- .25436265724+006
4	- .71956909266+004				- .1449495119+000
5	- .21GB366309+005				.1354036492+000
6	- .24692607345+005				.28893526596+000
7	.86941023862+005				.20162802300+000
8	- .77602100021+006				- .1075473712-002
9	- .22647938731+006				- .19929902668+000
10	.55919660039+006				- .3598111622+000
20	- .29725755313+011				
30	- .38462193224+013				- .1597433840+004
40	- .125533836132+022				- .7256142316+010
50	- .93220387212+032				- .4362393292+019
60	- .34819362850+042				- .36410666502+028
70	- .6613C52365+052				- .7773705775+030
80	- .5074606070+070				- .9237666352+049
90	- .13229674324+082				- .47227135338+051
91	- .24546193247+084				- .86104042316+073
92	- .46042417229+085				- .4546421556+075
93	- .8722999466+086				- .2895959732+076
94	- .1672016491+083				- .514750142014+077
95	- .32836591055+039				- .95462026324+079
96	- .6230037345+090				- .174993585515+080
97	- .12507226322+092				- .3390669125+081
98	- .34962322723+093				- .64920740091032
99	- .8032326162+094				- .12560717621624
100	- .1024565723+096				- .24554035004626
101	- .21065112460+097				- .436491432271+096
102	- .43702390933+093				- .53305904457+094
103	- .9166712693+099				- .5292391093+093
104	- .1939797577+101				- .5229660353-009
105	- .4143729296+102				- .2442372393-011
106	- .5934771120+103				- .1130096049-013
107	- .194436929+105				- .51010724254-016
108	- .42270513355+106				- .2353764293-013
109	- .9464323593+107				- .1059693035-020
110	- .2116320329+109				- .4723431415-023
111	- .4776034622+110				- .2091106739-025
112	- .46491432271+086				- .193296274+075

Table 11. Computation of $Y_r(10)$ from the JYIK equation with $x_1=10$, $x_2=1$, $i=0$, $m=100$, $j=3$.

r	$z(r)$	$(-)^{r_{K_r}(1)}$
0	(.43103242382+000)	
1	(-.00190723034+000)	
2	(-.162423332994+001)	
3	(-.71012623254+001)	
4	(.13121259924+000)	
5	(-.23050664924+002)	
6	(.470314332464+003)	
7	(-.91950602204+002)	
8	(.19102350164+011)	
9	(-.426014644334+012)	
10	(.101924360004+012)	
20	(.19131106244+029)	
30	(.35203237034+043)	
40	(.15520323842+065)	
50	(.76178056074+004)	
60	(.26032969494+105)	
70	(.6543633334+120)	
80	(.32052276624+148)	
90	(.77232124038+120)	
91	(-.14232056214+123)	
92	(.26372011354+125)	
93	(-.49316062214+127)	
94	(.94702056327+129)	
95	(-.10193900334+130)	
96	(.35336630332+132)	
97	(-.60237525164+135)	
98	(.15723111394+129)	
99	(-.27274233034+191)	
100	(.5553353194+193)	
		(.6294339364+028)
		(.4909146345224+029)
		(.11442430454+059)
		(.84063963544+013)
		(.266923635224+053)
		(.10064545004+120)
		(.63905439064+141)
		(.10439236294+164)
		(.13341463344+166)
		(.63722102744+163)
		(.614249004154+170)
		(.1429366344+173)
		(.21424030494+174)
		(.40313213494+172)
		(.22143443224+171)
		(.62143443224+172)
		(.16203442634+162)
		(.29290393244+162)
		(.590639031344136)

Table 12. Computation of $(-)^{r_{K_r}(1)}$ from the JYIK equation with $x_1=10$, $x_2=1$, $i=0$, $m=100$, $j=4$.

Example 4.1.3 In this example we compute the solutions $A_r^+(1)$, $J_r(1)$, $I_r(1)$, $Y_r(1)$ and $(-)^r K_r(1)$ of the homogeneous equation defined in Figure 6, with $x_1 = 1$, $x_2 = 1$, for $r = 0, 1, 2, \dots, 100$. Since each solution is monotonic (in magnitude) except possibly at the very beginning of the range of r , the algorithm is expected to be stable provided that the proper number of initial values is taken in each case. This number is $j = 1$ for $A_r(1)$, $j = 2$ for $J_r(1)$ and $I_r(1)$, and $j = 4$ for $Y_r(1)$ and $(-)^r K_r(1)$. However, in the case of $A_r^+(1)$, for reasons given earlier in this section, we take $j = 2$ instead of $j = 1$, thereby exchanging mild instability for a much improved rate of convergence.

Because of this mild instability, each individual rounding error, introduced at say $r = s$, is amplified in direct proportion to the number of steps taken beyond the point s , since in (4.1.7) $A_r/I_r = O(1/r)$. Thus, after 100 steps, approximately two decimal places of precision will be lost. On the other hand, if we were to take the number of initial conditions theoretically required (in this case it is $j=1$, since A_r is dominated by all of the other homogeneous solutions), then the convergence would be extremely slow: A terminal point of the order of 10^8 would be required to obtain 8 significant figures of accuracy in $A_{100}(1)$. Tables 13 through 17 give the results of these computations for selected values of r ; see §4.0 for a full description of these tables. Again, these results agree with the entries in Table 9.4 and 9.11 of [16]. In the case of $A_r^+(1)$, the method we have used is equivalent to computing $J_r(1)$ and $I_r(1)$ as type 2 solutions and then forming their difference.

[†]The function $A_r^+(1)$ is defined by $A_r^+(1) = J_r(1) - I_r(1)$; compare (4.1.6).

Table 13. Computation of $A_r(1) = J_r(1)-I_r(1)$ from the JYIK equation with $x_1=x_2=1$, $i=0$, $m=100$, $j=2$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$\eta_{r,1}(100)$	$J_r(1)$
0	(.7651976366+000) (-.4420505357+000)				
1	-.2755206999+001				
2	-.9373629133+000				
3	-.1977173463+000				
4	-.290455360-001				
5	-.356469756-002				
6	-.3364624510-033				
7	-.2713365326-004				
8	-.1319614390-005				
9	-.1457224939-006				
10					
20	-.6502469604-021				
30	-.1295626969-037				
40	-.7267926914-056				
50	-.2964124363-075				
60	-.1519300201-095				
70	-.1400320206-116				
80	-.2936532033-138				
90	-.1776296675-160				
91	-.997052453-163				
92	-.5541644712-165				
93	-.3044235047-167				
94	-.1654163362-169				
95	-.8921365003-172				
96	-.4723603633-174				
97	-.2483224227-176				
98	-.1295735942-178				
99	-.66723433073-181				
100	-.3406435317-183				
101	-.4720124310-185	.2451225522-006	.8431628772-100	.1043672361-196	
102	-.45992303974-183	.2379592624-004	.8431628763-102	.5065063535-197	
103	-.4256523083-180	.1121932340-010	.8431628783-103	.1402607093-205	
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Table 14. Computation of $J_r(1)$ from the JYIK equation with $x_1=x_2=1$, $i=0$, $m=100$, $j=2$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$\eta_{r,I}(100)$	$I_r(1)$
0	(-1266065379+100)				
1	(.36313916040+000)				
2	(-.32552004560+000)				
3	(-.10624243228+000)				
4	(-.21254450124000)				
5	(-.3235125121-001)				
6	(-.3726211111-002)				
7	(-.3501632211-003)				
8	(-.285365644-004)				
9	(-.1936492000-005)				
10	(-.121149325-006)				
20	(-.6354363643-021)				
30	(-.1616493605-032)				
40	(-.23521020231-050)				
50	(-.2936722032-070)				
60	(-.1532003356-095)				
70	(-.1410719255-116)				
80	(-.3036024662-133)				
90	(-.1726022005-150)				
91	(-.1000142810-162)				
92	(-.5521610561-165)				
93	(-.3056670229-167)				
94	(-.1632022036-169)				
95	(-.9939526574-172)				
96	(-.4762026063-174)				
97	(-.250100106-176)				
98	(-.1302496635-178)				
99	(-.6711115900-181)				
100	(-.34395628235-183)				
101	(-.1728032544-185)	.2451225522-006	.8473673991-123	.104391464-196	.4194266322-190
102	(-.3361354191-183)	.2376592624-004	.84736724001-123	.5939650706-192	.26563218236-192
103	(-.4276235442-190)	.1121932340-010	.8473674006-123	.1182703502-205	.06600060000
104					.0000000000

Table 15. Computation of $I_r(1)$ from the JYIK equation with $x_1=x_2=1$, $i=0$, $m=100$, $j=2$.

Table 16. Computation of $Y_r(1)$ from the JYLK equation with $x_1 = x_j = 1$, $i=0, m=100$, $j=4$.

r	$z(r)$	$(-)^{r_{K_r}(1)}$
0	(-.4210245025+000)	.629436926214020
1	(-.60193723003+000)	-.4763455284160
2	(-.10323212399+001)	-.36026105964000
3	(-.71013623253+001)	-.36526233431604
4	(-.6231241535+002)	-.442072020323005
5	(-.3609505396+003)	-.6236631433144006
6	(-.3652633313+004)	-.66046409940000
7	(-.4330702063+005)	-.18024233994000
8	(-.62235621231+006)	
9	(-.1000504092+006)	
10	(-.1337122392+009)	
20	(-.62943692363+023)	.10102359914164
30	(-.47861455239+040)	-.133441354824166
40	(-.1142220651+059)	-.33362162234165
50	(-.3496206050+070)	-.64424609464120
60	(-.79602315220+092)	-.444235365314123
70	(-.10064725500+120)	-.3143531964173
80	(-.53906499034141)	-.4031316224127
90	(-.10139263314164)	-.2336423224129
91	(-.18341254034166)	-.15263412394162
92	(-.33362102234163)	-.297963694241624
93	(-.6142490313+170)	.59003381354165
94	(-.11422363314173)	
95	(-.21430301964175)	
96	(-.40313216274177)	
97	(-.72364433224179)	
98	(-.15263117004182)	
99	(-.39796369424184)	
100	(-.59003381354186)	

Table 17. Computation of $(-)^{r_{K_r}(1)}$ from the JYIK equation with $x_1=x_2=1$, $i=0$, $m=100$, $j=4$.

4.2 Examples Involving the Inhomogeneous JYIK Equation

For fixed x and large r the Anger-Weber function has the asymptotic forms

$$(4.2.1) \quad E_{2r}(x) \sim \frac{2x}{(4r^2-1)\pi} , \quad E_{2r+1}(x) \sim \frac{2}{(2r+1)\pi} ;$$

compare [21]. Comparison of (4.2.1) with (4.1.1) and (4.1.2) shows that

$$(4.2.2) \quad \left\{ \begin{array}{l} I_r(x_2) \\ J_r(x_1) \end{array} \right\} < E_r(x) < \left\{ \begin{array}{l} (-)^r K_r(x_2) \\ Y_r(x_1) \end{array} \right\}, \quad r \rightarrow \infty,$$

for any given set of values of x_1 , x_2 , x . Thus $E_r(x)$ is a solution of type 2 of the first inhomogeneous equation defined in Figure 6 and the correct number of initial values of $E_r(x)$ to be used in every case is $j = 2$.

The asymptotic form of the Struve function, also taken from [21], is

$$(4.2.3) \quad H_r(x) \sim \frac{x}{\sqrt{2}\pi} \left(\frac{ex}{2r} \right)^r$$

for fixed x and large r . Comparison of (4.2.3) with (4.1.1) and (4.1.2) shows that

$$(4.2.4) \quad H_r(x) \sim \frac{x}{\sqrt{\pi r}} \left\{ \begin{array}{l} I_r(x) \\ J_r(x) \end{array} \right\}, \quad r \rightarrow \infty.$$

Thus $H_r(x)$ is a solution of type 1 of the second inhomogeneous equation defined in Figure 6 when $x_1 = x_2$; compare (4.1.8). However, since the separation of $H_r(x)$ from solutions of type 2 is even weaker than

that of the solution $A_r(x)$ defined by (4.1.6) it is again advantageous in most applications to specify the number of initial values appropriate to solutions of type 2, i.e., $j = 2$ instead of $j = 1$; compare Example 4.1.3.

Example 4.2.1 For the inhomogeneous fourth-order difference equations defined in Figure 6, we compute $E_r(1)$ with $x_1 = x_2 = 1$ for $r = 0, 1, 2, \dots, 100$; and $H_r(0.1)$ with $x_1 = x_2 = 0.1$, for $r = 0, 1, 2, \dots, 50$.[†] The appropriate number of initial values in these cases is $j = 2$ for $E_r(1)$ and $j = 1$ for $H_r(0.1)$. However, for reasons given above we actually use $j = 2$ for $H_r(0.1)$. Initial values for $E_r(1)$ and $H_r(0.1)$ were computed from series expansions given in [21] and checked against values found in [21] and [23]. Tables 18 and 19 give the results of these computations for selected values of r ; see §4.0 for a full description of these tables. They agree with the 10-figure values of $E_r(1)$ and $H_r(0.1)$ $r = 2, 3, 4, 5$, given in [23]; the 7 through 9-figure values of $E_r(1)$, $r = 2, 3, \dots, 10$, given in [21]; and the 9-figure values of $H_r(0.1)$, $r = 1, 2, \dots, 13$, given in [21]. In addition, they agree with series calculation of the functions to 10 significant figures in the case of $E_r(1)$, $r = 2, 3, \dots, 100$, and to 9 significant figures in the case of $H_r(0.1)$, $r = 2, 3, \dots, 50$. The loss of precision in the Struve function computation is due to mild instability which is acceptable in lieu of impossibly slow convergence.

[†]We restrict the sequence of $H_r(0.1)$ to $r \leq 50$ in order to avoid underflow in the computer. As suggested at the end of §4.0, this difficulty could be overcome by using the software described in [11].

r	$w_{100}(r)$	$y_r(100)$	$\eta_{r,1}(100)$	$E_r(1)$
0	(-.5636566270+000)			
1	(-.4331624362+000)			
2	-.403956112+001			
3	-.11784733198+002			
4	-.3753495822+001			
5	-.1597396355+002			
6	-.3140340313+001			
7	-.2676353332+002			
8	-.3947692529+001			
9	-.254956800+002			
10	-.284959721+001			
20	-.2634453606+001			
30	-.2635916191+001			
40	-.2612656172+001			
50	-.2599601067+001			
60	-.2599601423+001			
70	-.2539660036+001			
80	-.2573922335+001			
90	-.2573255665+001			
91	-.2542975365+003			
92	-.2574619393+001			
93	-.2393900229+003			
94	-.2574011042+001			
95	-.24246365293+003			
96	-.2573423056+001			
97	-.2494750555+003			
98	-.25722869631+001			
99	-.2546635994+003			
100	-.25722333697+001			
101	-.2597601595+003	.2451225522-006	.6367162628-004	.1576066173-002
102	-.25246190931001	.2379592624-004	.6362620235-004	.1514622220-003
103	-.2640522356+003	.1121932840-010	.6362471717-004	.7355123329-013
104	-.2531324556+001	.5447735345-009	.6362471725-004	.3467332779-013
105	-.2599463265+002	.370095333-015	.6362471728-004	.2477633711-017
106		.6362471728-004		.0000000000

Table 18. Computation of $E_r(1)$ from the JYIK equation with $x_1=x_2=1$, $i=0$, $m=100$, $j=2$.

r	$z(r)$	$w_{50}(r)$	$y_r(50)$	$\eta_{r,1}(50)$	$H_r(0.1)$
0	(.6359126999-001)				
1	(-.2120651601-002)				
2	(-.10181066632-006)				
3	(-.2909123739-003)				
4	(-.5307739737-010)				
5	(-.7347261321-012)				
6	(-.7912706103-014)				
7	(-.7053700528-016)				
8	(-.5319715276-018)				
9	(-.3499821171-020)				
10	(-.2036920172-022)				
11	(-.1062289307-024)				
12	(-.5024146393-027)				
13	(-.2170946455-029)				
14	(-.8537739352-032)				
15	(-.31334456090-034)				
16	(-.1093657399-036)				
17	(-.35153427053-039)				
18	(-.1061974447-041)				
19	(-.3025295937-044)				
20	(-.3155514110-047)				
25	(-.5553626624-050)				
30	(-.1323226920-073)				
35	(-.1339393907-097)				
40	(-.6462471933-102)				
41	(-.9175424095-105)				
42	(-.1003733294-107)				
43	(-.1214677153-110)				
44	(-.142326304-113)				
45	(-.1640339654-116)				
46	(-.1342346533-119)				
47	(-.202442771-122)				
48	(-.2173549674-125)				
49	(-.2389639923-128)				
50	(-.2359934415-131)				
51	(-.23316333159-134)	.1923246462-008	.2312680799-131	.4492442073-143	.2245322540-124
52	(-.2357376234-137)	.9071117562-006	.2312680799-131	.2096474211-143	.0000000000
53			.2312680799-131		.0000000000

Table 19. Computation of $H_r(0.1)$ from the JYIK equation with $x_1=x_2=0.1$, $i=0$, $m=50$, $j=2$.

4.3 Examples Involving the Homogeneous JYPQ Equation

For large values of r and fixed values of μ and x , the associated Legendre functions have the asymptotic forms

$$(4.3.1) \quad P_r^\mu(x) = \frac{\Gamma(r+\mu+1)}{\Gamma(r+3/2)} \left(\frac{1}{2} \pi \sin \theta\right)^{-1/2} \cos(r\theta + \frac{1}{2}\theta - \frac{1}{4}\pi + \frac{1}{2}\mu\pi) + O(\frac{1}{r})$$

and

$$(4.3.2) \quad Q_r^\mu(x) = \frac{\Gamma(r+\mu+1)}{\Gamma(r+3/2)} \left(\frac{\pi}{2 \sin \theta}\right)^{1/2} \cos(r\theta + \frac{1}{2}\theta + \frac{1}{4}\pi + \frac{1}{2}\mu\pi) + O(\frac{1}{r})$$

where $x = \cos \theta$ with $0 < \theta < \pi$; see [16, Chapter 8]. Consequently, for values of x less than 1 in magnitude, neither associated Legendre function dominates the other. Comparing (4.3.1) and (4.3.2) with (4.1.1) we obtain

$$(4.3.3) \quad J_r(x_1) < \begin{Bmatrix} P_r^\mu(x_2) \\ Q_r^\mu(x_2) \end{Bmatrix} < Y_r(x_1) , \quad r \rightarrow \infty$$

for all admissible values of μ , x_1 , x_2 . Thus the JYPQ equation has three distinct types of solution, and the dimensions of the subdominant subspaces of types 1, 2 and 3 are 1, 3 and 4 respectively; this contrasts with (4.1.3), (4.1.4), and (4.1.8).

Example 4.3.1 In this example we compute the solutions $J_r(1)$, $P_r(0.5)$, $Q_r(0.5)$ and $Y_r(1)$ of the homogeneous equation[†] defined in Figure 7, with $\mu = 0$, $x_1 = 1$, $x_2 = 0.5$, for $r = 0, 1, 2, \dots, 100$. Since these solutions exhibit their characteristic asymptotic separation, described by (4.3.3), from the very beginning of this range of r , the algorithm

[†]In accordance with custom, when the order of the associated Legendre functions is zero we drop the superscript in the notation.

will be stable provided the proper number of initial values is used in each case. This number is $j = 1$ for $J_r(1)$, $j = 3$ for $P_r(0.5)$ and $Q_r(0.5)$, and $j = 4$ for $Y_r(1)$.

Table 20 gives the numerical coefficients of the JYPQ operator with $\mu = 0$, $x_1 = 1$ and $x_2 = 0.5$ for $r = 0, 1, 2, \dots, 104$. Tables 21 and 22 give the numerical coefficients of the $(0,1)$ -factorization and the $(0,3)$ -factorization. Since the (i,j) -factorization $D_j^i = A_j^i B^{i+j}$ produced by the test program has the principal diagonal of A_j^i equal to $(1, 1, 1, \dots)$, we do not include values of $a_j(r)$.

Tables 23 through 26 give the results of the computation of the four desired solutions for selected values of r ; see §4.0 for a full description of these tables. Initial values were obtained from Table 9.4 of [16] in the case of the Bessel functions and by hand calculation of the explicit elementary forms given in [16], Section 8.4 in the case of the Legendre functions. The computed Bessel function values agree satisfactorily with the values given in [16]. The Legendre function values were compared with data available for $r \leq 10$ in [16] and (in the case of the $P_r(0.5)$) [17]. Agreement is satisfactory to the precision available in these references; this ranges from 6 to 8 significant figures. In addition, all of the computed Legendre function values agree satisfactorily with double-precision values computed by an unpublished program of Dr. J. M. Smith of the National Bureau of Standards. This is sufficient to demonstrate stability.

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
0	.2926250000+003	.3972500000+003	.2031250000+003	-.2300000000+002	-.5647500000+002
1	-.1137500000+003	-.6207500000+004	.5317875000+004	-.5647500000+003	-.3094500000+004
2	-.1931500000+004	-.3726675000+005	.3464662500+005	-.1049325000+005	-.1379233750+006
3	-.4499250000+004	-.1437410000+006	.4169141250+006	-.2759200000+006	-.4161117500+005
4	-.1379900000+005	-.4286322500+006	.1052276375+007	-.6161117500+005	-.1379233750+006
5	-.3422375000+005	-.1073633500+007	.2339131625+002	-.4226325000+006	-.42241332500+006
6	-.7358250000+005	-.21385320125+007	.472731630000+007	-.6339392125+007	-.6333920000+006
7	-.1426002500+006	-.4320093375+007	.65382322625+007	-.62220461500+006	-.62220461500+006
8	-.235654280000+006	-.81563822625+007	.14460355372+008	-.2614016062+003	-.9665065000+006
9	-.4306472500+006	-.92293470637+008	.9364194380+008	.42066336382+003	-.14464662350+006
10	-.6936350000+006	-.1363233321+009	.14366000592+009	.653463312+003	-.2106322000+006
11	-.1062391250+007	-.2432951512+003	.3946626637+003	.653463312+003	-.2980946750+007
12	-.1879629000+007	-.6145543962+003	.6562219425+008	.98428620482+003	-.62220461500+006
13	-.2239446750+007	-.92293470637+008	.9364194380+008	.1445344916+009	-.4125595500+007
14	-.3203796500+007	-.1363233321+009	.14366000592+009	.207236292344+009	-.5596929450+007
15	-.4441536250+007	-.1967361514+009	.207236292344+009	.2911715591+009	-.7430474000+007
16	-.5963220000+007	-.2771133526+009	.2911715591+009	.4016249729+009	-.9732060450+008
17	-.7924477250+008	-.3334169359+009	.4026264755+009	.5449469946+009	-.126659450+008
18	-.1066259550+008	-.5209222451+009	.5460469147+009	.723852365564+009	-.1614029425+008
19	-.1366649525+008	-.69723936043+009	.723852365564+009	.9609419921+009	-.20463517400+008
20	-.17289341604+008	-.9219321376+009	.96253671224+009	.1252059573+010	-.2554004375+008
21	-.2146333025+008	-.1203265039+010	.12539549223+010	.16131797222+010	-.3161744950+008
22	-.2675322650+008	-.15536329362+010	.16154162323+010	.2057457616+010	-.33798959325+008
23	-.3306099425+008	-.19337456381+010	.20592794563+010	.2698542069+010	-.42235359300+008
24	-.4048199400+008	-.25076393635+010	.2698542069+010	.32537257353+010	-.57630360254+008
25	-.4919264375+008	-.3143919729+010	.32537257353+010	.40445134239+010	-.6347091250+008
26	-.5934448750+008	-.3909116575+010	.40445134239+010	.4931064265+010	-.8163292325+008
27	-.74995254+008	-.46235496730+010	.4931064265+010	.6996543292+010	-.9674329200+008
28	-.8466099300+008	-.5902641263+010	.6996543292+010	.74127714333+010	-.11401163412+009
29	-.10011935422+009	-.7192339961+010	.74127714333+010	.8964532120+010	-.1336630555+009
30	-.1179833723+009	-.8693241160+010	.8964532120+010	.9957651896+010	-.1359342312+009
31	-.1381131762+009	-.10459666221+011	.1076966603+011	.1076196300+011	-.1359342312+009
32	-.16095263240+009	-.1260792932+011	.12467925842+011	.12359234366+011	-.13107743270+009
33	-.18166923915+009	-.14300920326+011	.1339632626+011	.162365366272+011	-.2093636402+009
34	-.24641822512+009	-.17616560274+011	.1609504431+011	.1808413365+011	-.34102772905+009
35	-.2834112950+009	-.207579280+011	.21295632422+011	.2496267940+011	-.2765191962+009
36	-.3247317727+009	-.2403414020+011	.2496267940+011	.3160123500+009	-.3160123500+009

Table 20. Numerical coefficients of the JYQ operator with $\mu=0$, $x_1=1$, and $x_2=0.5$.

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
27	-.3696107905+009	.2845505517+011	-.2947104125+011	-.2215610035+011	-.3593937157+009
33	-.492677938+009	.3311536389+011	-.33923007500+011	-.3391240794+011	-.400361713035+009
39	-.453410102004+009	.39336124621+011	-.39349952032+011	-.392976531+011	-.46222655392+009
40	-.51536365637+009	.44359999988+011	-.4539782103+011	-.4332335792+011	-.5215111210+010
41	-.68936362216+009	.5107652228+011	-.6224416964+011	-.52222230463+011	-.53650311637+009
42	-.672303008924+009	.5261983042+011	-.59929962776+011	-.5990595520+011	-.6502850045+009
43	-.75633693904+009	.6705940343+011	-.6353561287+011	-.6350937341+011	-.73656238136+009
44	-.820027245807+009	.7651031964+011	-.73147129084+011	-.70236036394+009	-.70236036394+009
45	-.9359653226+009	.8703851102+011	-.8836644367+011	-.914634212+011	-.914634212+011
46	-.10394363229+010	.9826026222+011	-.1027616380+012	-.1002220694+012	-.1012213162+010
47	-.11513919262+010	.1112212935+012	-.1139672269+012	-.1139301319+012	-.11226036349+010
48	-.1233356230+010	.1269944201+012	-.1235344295+012	-.12353446915+012	-.12353446915+012
49	-.14059572783+010	.1419374223+012	-.1427314506+012	-.14463303041+012	-.1372634257+010
50	-.1545283551+010	.1506093604+012	-.16253382654+012	-.1624823927+012	-.1618434410+010
51	-.1706993019+010	.1287926050+012	-.1321244212+012	-.1320244125+012	-.1670220006+010
52	-.1869233737+010	.1999911424+012	-.2036643161+012	-.20366904793+012	-.20339914632+010
53	-.2044233076+010	.2232372702+012	-.2232423620+012	-.22321846437+012	-.2010313236+010
54	-.2260622246+010	.2436939138+012	-.2530765900+012	-.2530144443+012	-.2199545364+010
55	-.24473113270+010	.2265241248+012	-.2813123626+012	-.2812621335+012	-.24036302394+010
56	-.26162323562+010	.2069024566+012	-.312212673231+012	-.31206353931+012	-.2632120325+010
57	-.2906504505+010	.240121256+012	-.34573032506+012	-.34562363661+012	-.23656216272+010
58	-.316023105+010	.32760463449+012	-.33232369180+012	-.3321632624+012	-.3106236334+010
59	-.3431711143+010	.4153030121+012	-.4219325551+012	-.4218353220+012	-.3524515234+010
60	-.37214736639+010	.5027106616+012	-.4330054103+012	-.4349124396+012	-.36604910367+010
61	-.4050492106+010	.5311601514+012	-.511601514+012	-.5115325145+012	-.3964523393+010
62	-.43597002024+010	.559564612+012	-.56219200436+012	-.56209236311+012	-.4290421439+010
63	-.4710076732+010	.6026625756+012	-.61630332235+012	-.61691616954+012	-.462365621333+010
64	-.5022024673+010	.6657341778+012	-.6257591552+012	-.62562015324+012	-.5603456533+010
65	-.5273300402+010	.7261235020+012	-.7393322000+012	-.7392109032+012	-.53994915274+010
66	-.8C9C4111216+010	.7962409535+012	-.8972363527+012	-.8972363527+012	-.8310463562+010
67	-.63433103502+010	.85908592634+012	-.9315162264+012	-.9315162264+012	-.62162343121+010
68	-.6515733723+010	.9423369334+012	-.96022615+012	-.96022615+012	-.9605224605+012
69	-.7315622162+010	.1031407610+013	-.1045733432+013	-.1045733432+013	-.7216234324+010
70	-.7820203326+010	.1121605401+013	-.1120323629+013	-.1120323629+013	-.7733924563+010
71	-.849230311+010	.1245303269+013	-.12343032705+013	-.12343032705+013	-.8239662731+010
72	-.8992304232+010	.1321806343+013	-.1324905321+013	-.1324905321+013	-.8809454032+010
73	-.9614900901+010	.14321553930+013	-.1451274195+013	-.1451274195+013	-.94855042224+010

Table 20. (Continued)

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
24	- . 10271573663+011	- . 15509233747+013	- . 1571147463+013	- . 1570939420+013	- . 1013465692+011
25	- . 1096361664+011	- . 169360329+013	- . 16935187+013	- . 1693716079+013	- . 163193410+011
26	- . 1169246276+011	- . 181254379+013	- . 183562875+013	- . 183503171+013	- . 1154064910+011
27	- . 1245956198+011	- . 19566702715+013	- . 1930599428+013	- . 1930356302+013	- . 1229956343+011
28	- . 1326640672+011	- . 2109273715+013	- . 2136395979+013	- . 2135141033+013	- . 1309851563+011
29	- . 141451566+011	- . 2272347769+013	- . 2300145346+013	- . 2299872531+013	- . 1393842354+011
30	- . 1500545361+011	- . 2445795237+013	- . 2475352993+013	- . 247507103+013	- . 1493024144+011
31	- . 1594032261+011	- . 2639137415+013	- . 2661543413+013	- . 2661243403+013	- . 1574646367+011
32	- . 1692226406+011	- . 2825915003+013	- . 2949260556+013	- . 2853951192+013	- . 1671232327+011
33	- . 1796143930+011	- . 30386324562+013	- . 3069063350+013	- . 306324126+013	- . 1773272943+011
34	- . 1903013029+011	- . 3254039095+013	- . 3291531161+013	- . 3291241542+013	- . 1883962907+011
35	- . 2016003948+011	- . 3437568655+013	- . 35227314235+013	- . 3526953730+013	- . 1992566553+011
36	- . 213439401+011	- . 3734499404+013	- . 37766904432+013	- . 3776619400+013	- . 2109771524+011
37	- . 2256003073+011	- . 3996677370+013	- . 4041210260+013	- . 4040320134+013	- . 22332422793+011
38	- . 2387544203+011	- . 4220662091+013	- . 4320662091+013	- . 4320256146+013	- . 23607232313+011
39	- . 2592267203+011	- . 4566266334+013	- . 4616036265+013	- . 4615611239+013	- . 2494615033+011
40	- . 26642272650+011	- . 4825230103+013	- . 4930042633+013	- . 4922604075+013	- . 26350044952+011
91	- . 2911945021+011	- . 5201949012+013	- . 52367439215+015	- . 5236973946+013	- . 2294363756+011
92	- . 2966093033+011	- . 5546436613+013	- . 5604976211+013	- . 5604922652+013	- . 2934336539+011
93	- . 3126928792+011	- . 5909727243+013	- . 5921449321+013	- . 592094763+013	- . 3093660013+011
94	- . 3294664250+011	- . 6392634536+013	- . 6357676321+013	- . 6357676321+013	- . 32599490593+011
95	- . 3269522692+011	- . 6694996304+013	- . 6264499743+013	- . 6263952303+013	- . 3433373426+011
96	- . 3651727046+011	- . 7120677662+013	- . 7192217746+013	- . 7192217746+013	- . 3614606132+011
97	- . 3841506390+011	- . 7562570692+013	- . 7643838406+013	- . 7643834629+013	- . 3402313230+011
98	- . 4039096043+011	- . 8037995480+013	- . 8117375217+013	- . 8117375217+013	- . 39963303092+011
99	- . 42242231052+011	- . 8561699190+013	- . 8616551304+013	- . 8616903443+013	- . 4202290134+011
100	- . 4450661399+011	- . 9050139631+013	- . 9130940423+013	- . 9130299909+013	- . 4414612233+011
101	- . 4661429001+011	- . 9596012842+013	- . 9600572850+013	- . 9602862925+013	- . 4636524194+011
102	- . 4912869272+011	- . 1016340512+014	- . 1026542093+014	- . 1026420936+014	- . 4664202632+011
103	- . 5152200227+011	- . 1026399353+014	- . 1067072959+014	- . 1065992930+014	- . 5103161692+011
104	- . 5403823030+011	- . 1139364632+014	- . 1150464230+014	- . 1150464230+014	- . 53500331052+011

Table 20. (Concluded) .

r	$a_0(r)$	$b_0(r+1)$	$b_1(r+1)$	$b_2(r+1)$	$b_3(r+1)$
0	- .11373309000+000	.2932350000+000	- .3972350000+000	.2031250000+000	- .2300000000+000
1	- .35246394214e01	.34935722962e004	- .5491403344e064	.5296890246e004	- .56327510000e003
2	- .12373303220e001	.23633303220e005	- .3046245312e+005	.3391415123e+006	- .36943503220e-005
3	- .5820203926e000	.1045436939e006	- .1239321237e+006	.1361169123e+006	- .10460345600e+005
4	- .3574112619e000	.32224520351504e006	- .38876646674e+006	.41424263292e+006	- .22598320000e+005
5	- .23146536769e000	.35498791504e006	- .2814626902e+006	.1046664324e+006	- .616144000e+005
6	- .16439276614e000	.1966693203e002	- .2195025364e602	.2839022293e+002	- .42263246000e+006
7	- .12994663124e000	.4052125329e+002	- .44705050504e+002	.2705251992e+002	- .223412829500e+006
8	- .10363235924e000	.76036417264e+002	- .334160163294e+002	.63160320311e+002	- .63339406000e+006
9	- .7763295203e000	.1370439101e+002	- .14901514433e+002	.14545265663e+002	- .63220462500e+006
10	- .71503399569e-001	.2322462203e003	- .29369263046e+003	.26691310692e+003	- .96360040000e+006
11	- .6793921597e-001	.3770043613e+003	- .40102923502e+003	.42000553223e+003	- .144359162910e+006
12	- .60264747426e-001	.5902647616e+003	- .63303763656e+003	.663603030365e+003	- .2105372360016e006
13	- .543333511257e-001	.8950392719e+003	- .936938038916e+003	.93696412923e+003	- .39806934575010e006
14	- .49363043652e-001	.132113390124e+003	- .1401440126e+009	.1443873202e+009	- .4125875650010e005
15	- .4512909245e-001	.1904354122e+003	- .2018205292e+009	.207020655804e+009	- .535693925016e005
16	- .4164460305e-001	.2667354463e+003	- .281329222122e+009	.29038366432e+009	- .246024500090e005
17	- .3935392612e-001	.3221922513e+003	- .39130969922e+009	.4013320505e+009	- .97360026000e005
18	- .35913303891e-001	.50563241323e+003	- .53162523674e+009	.64459542413e+009	- .426653945010e005
19	- .32595322950e-001	.67939638263e+003	- .7113633352e+009	.72810313954e+009	- .46120294265e+006
20	- .31334555639e-001	.89953249223e+003	- .93956412402e+009	.9604314323e+009	- .30435123300e+003
21	- .2974120412e-001	.1179321341e+010	- .12353905354e+010	.12614541506e+010	- .255540134325e+006
22	- .281263151621e-001	.15181194091e+010	- .16302244374e+010	.1612461506e+010	- .6161424496010e003
23	- .26664572240e-001	.194666096724e+010	- .20167638394e+010	.2066314553e+010	- .467939392510e003
24	- .2524903656e-001	.2486635046e+010	- .2549421233e+010	.2592550354e+010	- .62256292000e+006
25	- .2416509224e-001	.3023330242e+010	- .3194513033e+010	.3162522354e+010	- .16203366664e003
26	- .23065025914e-001	.3331429511e+010	- .3920102323e+010	.4039230323e+010	- .63242613450e+003
27	- .22073270826e-001	.47359262264e+010	- .48963795994e+010	.4929501364e+010	- .61632433264e+003
28	- .211606021e-001	.5226042032e+010	- .59965371524e+010	.6094316254e+010	- .962439920010e003
29	- .2021310243e-001	.6730441631e+010	- .72936393435e+010	.741033123e+010	- .144913614710e003
30	- .195262339e-001	.8556340212e+010	- .86116763014e+010	.8955424723e+010	- .13366403956e+003
31	- .1831460241e-001	.10293730164e+011	- .1060119504e+011	.1025933353e+011	- .15593430125e+003
32	- .1844002026e-001	.12315632943e+011	- .1267222294e+011	.1265644004e+011	- .1810763226e+003
33	- .1731433029e-001	.14653050424e+011	- .16020114532e+011	.16233305533e+011	- .20936342124e+003
34	- .1667333650e-001	.17350450694e+011	- .17350450694e+011	.17350450694e+011	- .246443637224e+003
35	- .1653271829e-001	.2045521204e+011	- .2101126514e+011	.2129167424e+011	- .2775121942e+003
36	- .1531051020e-001	.2403069373e+011	- .246443637224e+011	.2496437969e+011	- .3160136500e+003

Table 21. Numerical coefficients of the (0,1)-factorization of the JYQ operator with $\mu=0$, $x_1=1$ and $x_2=0.5$.

r	$a_0(r)$	$b_0(r+1)$	$b_1(r+1)$	$b_2(r+1)$	$b_3(r+1)$
36	- .15227182315-000	.28075603220-000	.0+010403257665.	.-28729690220+010	.0+13236331336-3356.
37	- .1493604524-001	.32269592245-001	.0+03264553220-001	.-33949395322+001	.0+013236331336-3356.
38	- .1450421562-001	.32910314562-001	.0+03264553220+001	.-33929433949+001	.0+013236331336-3356.
39	- .1410696923-001	.33312572646+001	.0+03264553220+001	.-43324131294+01	.0+013236331336-3356.
40	- .13217207923-001	.30441390327-001	.0+03264553220+001	.-51621404111+01	.0+013236331336-3356.
41	- .1335442956-001	.57923266514+01	.0+03264553220+001	.-5923266514+01	.0+013236331336-3356.
42	- .1301020412-001	.66293727436+01	.0+03264553220+001	.-67756325560+01	.0+013236331336-3356.
43	- .1265816219-001	.75569955604+01	.0+03264553220+001	.-73656923572+01	.0+013236331336-3356.
44	- .1232302735-001	.8224242150+01	.0+03264553220+001	.-82220139014+01	.0+013236331336-3356.
45	- .119727318-001	.90462499194+01	.0+03264553220+001	.-91562312121+01	.0+013236331336-3356.
46	- .11793144632-001	.1105456499+012	.0+03264553220+001	.-11272724595+012	.0+013236331336-3356.
47	- .1152348162-001	.1242993162+012	.0+03264553220+001	.-1222216921+012	.0+013236331336-3356.
48	- .1126570423-001	.14056326120+012	.0+03264553220+001	.-14022834506+012	.0+013236331336-3356.
49	- .110101013573-001	.1572306933+012	.0+03264553220+001	.-1609395729+012	.0+013236331336-3356.
50	- .1072316636-001	.1720621625+012	.0+03264553220+001	.-1803724569+012	.0+013236331336-3356.
51	- .1055699116-001	.1900869520+012	.0+03264553220+001	.-2017223330+012	.0+013236331336-3356.
52	- .1034905990-001	.2211520494+012	.0+03264553220+001	.-22251875725+012	.0+013236331336-3356.
53	- .1013181773-001	.2664120605+012	.0+03264553220+001	.-2507749967+012	.0+013236331336-3356.
54	- .9931252479-002	.2746334934+012	.0+03264553220+001	.-27363091910+012	.0+013236331336-3356.
55	- .97394072323-002	.3093873225+012	.0+03264553220+001	.-31203192331+012	.0+013236331336-3356.
56	- .9554331036-002	.3305619164+012	.0+03264553220+001	.-3427219935+012	.0+013236331336-3356.
57	- .9326127674-002	.3728229390+012	.0+03264553220+001	.-3769965416+012	.0+013236331336-3356.
58	- .9204420476-002	.4112955639+012	.0+03264553220+001	.-4164452936+012	.0+013236331336-3356.
59	- .9030362666-002	.4539236633+012	.0+03264553220+001	.-461129266750+012	.0+013236331336-3356.
60	- .8879430923-002	.4996305240+012	.0+03264553220+001	.-50255326663+012	.0+013236331336-3356.
61	- .87294024932-002	.5493405243+012	.0+03264553220+001	.-5577234033+012	.0+013236331336-3356.
62	- .8675961222-002	.6032794635+012	.0+03264553220+001	.-614938334+012	.0+013236331336-3356.
63	- .8504420476-002	.6606239466+012	.0+03264553220+001	.-6705535334+012	.0+013236331336-3356.
64	- .83461930452-002	.7220317344+012	.0+03264553220+001	.-7332346263+012	.0+013236331336-3356.
65	- .8193736665-002	.7902561460+012	.0+03264553220+001	.-80172932463+012	.0+013236331336-3356.
66	- .8058900562-002	.8526194211+012	.0+03264553220+001	.-860323062+012	.0+013236331336-3356.
67	- .7927557443-002	.9404281129+012	.0+03264553220+001	.-9592641193+012	.0+013236331336-3356.
68	- .7901205935-002	.1023932510+012	.0+03264553220+001	.-10333117672+012	.0+013236331336-3356.
69	- .7773252195-002	.1113651923+013	.0+03264553220+001	.-1139014310+013	.0+013236331336-3356.
70	- .76660333533-002	.1209264424+013	.0+03264553220+001	.-1226235696+013	.0+013236331336-3356.
71	- .7543666936-002	.1312694633+013	.0+03264553220+001	.-1339241562+013	.0+013236331336-3356.
72	- .7438103545-002	.1423414925+013	.0+03264553220+001	.-1451209332+013	.0+013236331336-3356.
73	- .7326592002-002	.1533414925+013	.0+03264553220+001	.-1563603824+013	.0+013236331336-3356.

Table 21. (Continued)

r	$a_0(r)$	$b_0(r+1)$	$b_1(r+1)$	$b_2(r+1)$	$b_3(r+1)$
74	-7219193799-002	1540531103+013	156067020946+013	157020370946+013	-101326563634+011
75	-71161774235-003	16662533331+013	1682755634+013	1710+2034013	-1013193534+011
76	-701252177235-003	16604113330+013	16804020202+013	1710+2034013	-1013193534+011
77	-693056977793-002	19434515+013	196790020013	213505036+013	-10131440033+012
78	-6834637035-002	2095340419+013	21213826220+013	22352660225+013	-1024502034+012
79	-66951635102+002	2450606650+013	24600702022+013	25453311154+013	-1024393653+012
80	-6633193533+013	26130092004+013	26453311154+013	27124525225+013	-1024393653+012
81	-6562423543+013	286243004+013	29224525225+013	2996296225+013	-1024393653+012
82	-6499362430+013	310+2034013	314+2253502+013	32124525225+013	-1024393653+012
83	-64363543+013	33623350+013	34224525225+013	3496296225+013	-1024393653+012
84	-63733543+013	36223350+013	36824525225+013	3756296225+013	-1024393653+012
85	-63103543+013	38823350+013	39424525225+013	4016296225+013	-1024393653+012
86	-62473543+013	41423350+013	42024525225+013	4276296225+013	-1024393653+012
87	-61843543+013	44023350+013	44624525225+013	4536296225+013	-1024393653+012
88	-61213543+013	46623350+013	47224525225+013	4796296225+013	-1024393653+012
89	-60583543+013	49223350+013	49824525225+013	5056296225+013	-1024393653+012
90	-59953543+013	51823350+013	52424525225+013	5316296225+013	-1024393653+012
91	-59323543+013	54423350+013	55024525225+013	5576296225+013	-1024393653+012
92	-58693543+013	57023350+013	57624525225+013	5836296225+013	-1024393653+012
93	-58063543+013	59623350+013	60224525225+013	6096296225+013	-1024393653+012
94	-57433543+013	62223350+013	62824525225+013	6356296225+013	-1024393653+012
95	-56803543+013	64823350+013	65424525225+013	6616296225+013	-1024393653+012
96	-56173543+013	67423350+013	68024525225+013	6876296225+013	-1024393653+012
97	-55543543+013	70023350+013	70624525225+013	7136296225+013	-1024393653+012
98	-54913543+013	72623350+013	73224525225+013	7396296225+013	-1024393653+012
99	-54283543+013	75223350+013	75824525225+013	7656296225+013	-1024393653+012
100	-53653543+013	77823350+013	78424525225+013	7916296225+013	-1024393653+012
101	-530234009-9302	80423350+013	81024525225+013	8176296225+013	-1024393653+012
102	-523923556-002	83023350+013	83624525225+013	8436296225+013	-1024393653+012
103	-517623551-002	85623350+013	86224525225+013	8696296225+013	-1024393653+012
104	-511323244+012	88223350+013	88824525225+013	8956296225+013	-1024393653+012
105	-505023405-002	90823350+013	91424525225+013	9216296225+013	-1024393653+012
106	-508723445-002	93423350+013	94024525225+013	9476296225+013	-1024393653+012
107	-502423446-002	96023350+013	96624525225+013	9736296225+013	-1024393653+012
108	-506123447-002	98623350+013	99224525225+013	10000296225+013	-1024393653+012
109	-509823448-002	10123350+013	101824525225+013	102600296225+013	-1024393653+012
110	-503523449-002	10383350+013	104424525225+013	105200596225+013	-1024393653+012
111	-507223450-002	10643350+013	107024525225+013	107800896225+013	-1024393653+012
112	-510923451-002	10903350+013	109624525225+013	110401196225+013	-1024393653+012
113	-5146011653-002	11163350+013	112224525225+013	113001496225+013	-1024393653+012
114	-5183411055-002	11423350+013	114824525225+013	115601796225+013	-1024393653+012
115	-5219193799-002	11683350+013	117424525225+013	118202096225+013	-1024393653+012
116	-525523556-002	11943350+013	120024525225+013	120802396225+013	-1024393653+012
117	-529223557-002	12203350+013	122624525225+013	123402686225+013	-1024393653+012
118	-532923558-002	12463350+013	125224525225+013	126002976225+013	-1024393653+012
119	-536623559-002	12723350+013	127824525225+013	128603306225+013	-1024393653+012
120	-540323560-002	13083350+013	131424525225+013	132203546225+013	-1024393653+012
121	-543923561-002	13343350+013	134024525225+013	134803826225+013	-1024393653+012
122	-547623562-002	13603350+013	136624525225+013	137404106225+013	-1024393653+012
123	-551323563-002	13863350+013	139224525225+013	140004386225+013	-1024393653+012
124	-555023564-002	14123350+013	141824525225+013	142604666225+013	-1024393653+012
125	-558723565-002	14383350+013	144424525225+013	145204946225+013	-1024393653+012
126	-562423566-002	14643350+013	147024525225+013	147805326225+013	-1024393653+012
127	-566123567-002	14903350+013	149624525225+013	150405696225+013	-1024393653+012
128	-569823568-002	15163350+013	152224525225+013	153006076225+013	-1024393653+012
129	-573523569-002	15423350+013	154824525225+013	155606456225+013	-1024393653+012
130	-577223570-002	15683350+013	157424525225+013	158206836225+013	-1024393653+012
131	-580923571-002	15943350+013	160024525225+013	160807216225+013	-1024393653+012
132	-584623572-002	16203350+013	162624525225+013	163407596225+013	-1024393653+012
133	-588323573-002	16463350+013	165224525225+013	166008076225+013	-1024393653+012
134	-592023574-002	16723350+013	167824525225+013	168608456225+013	-1024393653+012
135	-595723575-002	17083350+013	171424525225+013	172208836225+013	-1024393653+012
136	-599423576-002	17343350+013	174024525225+013	174809216225+013	-1024393653+012
137	-603123577-002	17603350+013	176624525225+013	177409596225+013	-1024393653+012
138	-606823578-002	17863350+013	179224525225+013	180010976225+013	-1024393653+012
139	-610523579-002	18123350+013	181824525225+013	182601356225+013	-1024393653+012
140	-614223580-002	18383350+013	184424525225+013	185201736225+013	-1024393653+012
141	-617923581-002	18643350+013	187024525225+013	187802116225+013	-1024393653+012
142	-621623582-002	18903350+013	189624525225+013	190402496225+013	-1024393653+012
143	-625323583-002	19163350+013	192224525225+013	193002876225+013	-1024393653+012
144	-629023584-002	19423350+013	194824525225+013	195603256225+013	-1024393653+012
145	-632723585-002	19683350+013	197424525225+013	198203636225+013	-1024393653+012
146	-636423586-002	19943350+013	200024525225+013	200804016225+013	-1024393653+012
147	-640123587-002	20203350+013	202624525225+013	203404396225+013	-1024393653+012
148	-643823588-002	20463350+013	205224525225+013	206004776225+013	-1024393653+012
149	-647523589-002	20723350+013	207824525225+013	208605156225+013	-1024393653+012
150	-651223590-002	21083350+013	211424525225+013	212205536225+013	-1024393653+012
151	-654923591-002	21343350+013	214024525225+013	214805916225+013	-1024393653+012
152	-658623592-002	21603350+013	216624525225+013	217406296225+013	-1024393653+012
153	-662323593-002	21863350+013	219224525225+013	220006676225+013	-1024393653+012
154	-666023594-002	22123350+013	221824525225+013	222607056225+013	-1024393653+012
155	-670723595-002	22383350+013	224424525225+013	225207436225+013	-1024393653+012
156	-674423596-002	22643350+013	227024525225+013	227807816225+013	-1024393653+012
157	-678123597-002	22903350+013	229624525225+013	230408196225+013	-1024393653+012
158	-681823598-002	23163350+013	232224525225+013	233008576225+013	-1024393653+012
159	-685523599-002	23423350+013	234824525225+013	235608956225+013	-1024393653+012
160	-689223600-002	23683350+013	237424525225+013	238209336225+013	-1024393653+012
161	-692923601-002	23943350+013	240024525225+013	240809716225+013	-1024393653+012
162	-696623602-002	24203350+013	242624525225+013	243409096225+013	-1024393653+012
163	-700323603-002	24463350+013	245224525225+013	246009476225+013	-1024393653+012
164	-704023604-002	24723350+013	247824525225+013	248609856225+013	-1024393653+012
165	-707723605-002	25083350+013	251424525225+013	252209836225+013	-1024393653+012
166	-711423606-002	25343350+013	254024525225+013	254809816225+013	-1024393653+012
167	-715123607-002	25603350+013	256624525225+013	257409896225+013	-1024393653+012
168	-718823608-002	25863350+013	259224525225+013	260009876225+013	-1024393653+012
169	-722523609-002	26123350+013	261824525225+013	262609856225+013	-1024393653+012
170	-726223610-002	26383350+013	264424525225+013	265209836225+013	-1024393653+012
171	-730023611-002	26643350+013	267024525225+013	267809816225+013	-1024393653+012
172	-733723612-002	26903350+013	269624525225+013	270409896225+013	-1024393653+012
173	-737423613-002	27163350+013	272224525225+013	273009876225+013	-1024393653+012
174	-741123614-002	27423350+013	274824525225+013	275609856225+013	-1024393653+012
175	-744823615-002	27683350+013	277424525225+013	278209836225+013	-1024393653+012
176	-748523616-002	27943350+013	280024525225+013	280809816225+013	-1024393653+012
177	-752223617-002	28203350+013	282624525225+013	283409896225+013	-1024393653+012
178	-755923618-002	28463350+013	285224525225+013	286009876225+013	-1024393653+012
179	-759623619-002	28723350+013	287824525225+013	288609856225+013	-1024393653+012
180	-763323620-002	29163350+013	292224525225+013	293009836225+013	-1024393653+012
181	-767023621-002	29423350+013	294824525225+013	295609816225+013	-1024393653+012
182	-770723622-002	29683350+013	297424525225+013	298209896225+013	-1024393653+012
183	-774423623-002	30043350+013	301024525225+013	301809876225+013	-1024393653+012
184	-778123624-002	30303350+013	303624525225+013	304409856225+013	-1024393653+012
185	-781823625-002	30563350+013	306224525225+013	307009836225+013	-1024393653+012
186	-785523626-002</td				

Table 21. (Concluded).

r	$a_0(r)$	$a_1(r)$	$a_2(r)$	$b_0(r+3)$	$b_1(r+3)$
0	-1137500000+003	.2926250000+003	-.3972500000+003	.2031250000+003	-.23000000000+002
1	-.1031500000+004	.4393325000+004	-.2056000000+002	.4614995000+004	-.563750000+003
2	-.4499250000+004	.1511243077+003	-.7302926499+001	.3043964450+005	-.3094560000+004
3	-.67931353046+002	.261693M329+002	-.4226250523+001	.1243452428+006	-.104805000+005
4	-.7416356723+001	.1203972078+002	-.2531241331+001	.3340415685+003	-.275950000+005
5	-.241936029+001	.7549424245+001	-.2537695324+001	.9313616523+006	-.6161725000+005
6	-.1142456422+001	.5511216766+001	-.2262642195+001	.2199724972+007	-.1226375000+006
7	-.66554772451+000	.4391577324+001	-.2043651722+001	.4420169592+007	-.2241332500+006
8	-.4035367556+000	.3696733496+001	-.13936665124+001	.6415510563+002	-.3438970000+003
9	-.31326000214+000	.3226353226+001	-.1778953183+001	.1490078256+003	-.6220467500+006
10	-.23726196994+000	.2890343365+001	-.1690223036+001	.2603326356+003	-.2665045000+006
11	-.1875231790+000	.2639729169+001	-.1619626501+001	.4050092735+003	-.1443466230+007
12	-.1529316090+009	.2445737009+001	-.1562021342+001	.6303536455+003	-.2166322000+007
13	-.1273835352+000	.2291576720+001	-.1513166244+001	.9290730055+003	-.3936945250+007
14	-.1090950174+000	.2166235100+001	-.1473295322+001	.14014111875+009	-.412650500+007
15	-.9460421126+001	.2062021339+001	-.1439277994+001	.2013250455+009	-.5596936240+007
16	-.8316191206+001	.1925613011+001	-.1439432591+001	.286238005+009	-.7460275000+007
17	-.7394398399+001	.1901457029+001	-.1363373044+001	.3943046345+009	-.9289050240+007
18	-.6639261009+001	.1837531347+001	-.1360423582+001	.5316297255+009	-.126539450+006
19	-.6011413303+001	.1732015796+001	-.1330069944+001	.7115562675+009	-.1612939425+003
20	-.5482356641+001	.1733256043+001	-.1321669237+001	.9395535005+009	-.2042517360+003
21	-.5032304481+001	.1690132144+001	-.13065521397+001	.1225321001+000	-.2554024375+003
22	-.4644353361+001	.1651743531+001	-.12209751699+001	.1630231340+000	-.31617424980+003
23	-.4303941179+001	.1617341179+001	-.1227341262+001	.2016771727+000	-.3320982355+003
24	-.4014772434+001	.1563352305+001	-.122631113474+001	.2549457023+000	-.45222389300+003
25	-.3755231066+001	.156429923116+001	-.1253912353+001	.3194497021+000	-.5206346625+003
26	-.3539413300+001	.1532763668+001	-.124336120746+001	.3920039436+000	-.6442091425+003
27	-.3290363295+001	.1509465015+001	-.1224122395+001	.40965459400+000	-.6163232323+003
28	-.3136441631+001	.1481090136+001	-.1226343953+001	.5996514641+010	-.9274339200+003
29	-.2970665241+001	.1460422073+001	-.1217292515+001	.72295014438+010	-.114013341420+009
30	-.2820616784+001	.1450265561+001	-.1209623874+001	.8319231636+010	-.133664034540+009
31	-.2634053322+001	.1433453322+001	-.12035655723+001	.1060116453+011	-.1559342342+009
32	-.259657425+001	.141728310+001	-.1195943209+001	.1362324939+011	-.16102414861+014
33	-.2445643304+001	.1403214520+001	-.1189757762+001	.1607414861+014	-.2093646472+009
34	-.2340858309+001	.1389749926+001	-.1183946134+001	.1783621956+011	-.2410273946+009
35	-.2243274594+001	.1377055323+001	-.1178429611+001	.2101172432+011	-.2265191943+009
36	-.2154994421+001	.1365475743+001	-.1173223563+001	.2465443010+011	-.3160123500+009

Table 22. Numerical coefficients of the (0, 3)-factorization of the JYPQ operator with $\mu=0$, $x_1=1$ and $x_2=0.5$.

r	$a_0(r)$	$a_1(r)$	$b_0(r+3)$	$b_1(r+3)$
37	- . 3022043042-001	- . 1352003907+001	- . 28726350366+011	- . 3599331574+009
38	- . 1953051946-001	- . 11634653959+001	- . 4035433955-010	- . 6034330326+011
39	- . 1203235000-001	- . 0359666324+001	- . 3249303036+011	- . 4623536224+011
40	- . 1876032020-1-001	- . 1323243365+001	- . 4464230024+011	- . 6314511336+011
41	- . 1294034240-0-01	- . 1153037342+001	- . 5162123155+011	- . 6666630113+011
42	- . 1785534202-0-01	- . 131363996320+001	- . 5923263896+011	- . 6582220000+011
43	- . 1626263592-0-01	- . 1307022928+001	- . 6226622435+011	- . 2365663000+011
44	- . 1626261926-0-01	- . 1292077664+001	- . 6226622435+011	- . 6582220000+011
45	- . 1600186400-001	- . 1291531629+001	- . 6226622435+011	- . 6582220000+011
46	- . 1836529631-001	- . 1294935562+001	- . 6226622435+011	- . 6582220000+011
47	- . 183650502-0-01	- . 1272633021+001	- . 6249000205+001	- . 6666630000+011
48	- . 1290527241-0-01	- . 1271600000+001	- . 6322693634+011	- . 4222693634+011
49	- . 1290527241-0-01	- . 1271600000+001	- . 6322693634+011	- . 4222693634+011
50	- . 14103439470-001	- . 12352056945+001	- . 126928628+001	- . 13022336369+012
51	- . 1628529631-001	- . 12531723952+001	- . 123432056+001	- . 1609394453+012
52	- . 1628527241-001	- . 1247243942+001	- . 120950374+001	- . 1303223221+012
53	- . 1204549245-001	- . 1225536696+001	- . 1145561902+001	- . 2017221901+012
54	- . 1204549245-001	- . 1225536696+001	- . 1145561902+001	- . 2017221901+012
55	- . 1242243290-0-01	- . 1232750663+001	- . 1140566214+001	- . 2402243062+012
56	- . 1242243290-0-01	- . 1232750663+001	- . 1140566214+001	- . 2402243062+012
57	- . 11545454250-0-01	- . 1232221242+001	- . 1092602324+001	- . 2622432325+012
58	- . 11545454250-0-01	- . 1232221242+001	- . 1092602324+001	- . 2622432325+012
59	- . 1108194503-0-01	- . 1214516226+001	- . 104153000+001	- . 3106343236+012
60	- . 1023632012-0-01	- . 12026092760+001	- . 102263000+001	- . 3249363236+012
61	- . 1063466420-0-01	- . 1202323212+001	- . 1006695650+001	- . 3660344656+012
62	- . 104153000+001	- . 1202323212+001	- . 1006695650+001	- . 3660344656+012
63	- . 104153000+001	- . 1202323212+001	- . 1006695650+001	- . 3660344656+012
64	- . 1021632320-0-01	- . 1196327266+001	- . 102363000+001	- . 3636363236+012
65	- . 1001632362-0-01	- . 1196534250+001	- . 10926243232+001	- . 3600446456+012
66	- . 0663160045-0-02	- . 1190457295+001	- . 10926243232+001	- . 3699536746+012
67	- . 0663160045-0-02	- . 1190457295+001	- . 10926243232+001	- . 3699536746+012
68	- . 0460439615-0-02	- . 1183633960+001	- . 1091461463+001	- . 3203436369+012
69	- . 9286102522-0-02	- . 1181460641+001	- . 1090032542+001	- . 3216924322+012
70	- . 9123705221-0-02	- . 1182624100+001	- . 10837566231+001	- . 3216924322+012
71	- . 0363294250-0-02	- . 1179194299+001	- . 1082462239+001	- . 3216924322+012
72	- . 0363294250-0-02	- . 1179194299+001	- . 1082462239+001	- . 3216924322+012
73	- . 0516118622-0-02	- . 1163321315+001	- . 10837566231+001	- . 3216924322+012

Table 22. (Continued)

r	$a_0(r)$	$a_1(r)$	$a_2(r)$	$b_0(r+3)$	$b_1(r+3)$
74	- .85726531693-002	.11657925559+001	- .1022644943+001	.1560670497+013	- .10134656934011
75	- .82413279439-002	.11634401439+001	- .10316322039+001	.1637253221+013	- .1031932410411
76	- .8410306091-002	.1161163263+001	- .1030429709+001	.16233441576+013	- .11540649101611
77	- .79334663343-002	.1559229791+001	- .10793660231+001	.1967900293+013	- .12299656043+011
78	- .7859335565-002	.1557265664+001	- .1078330169+001	.2121877720+013	- .1309251590+011
79	- .77400935922-002	.154661360+001	- .1077320379+001	.2235766226+013	- .1393612353+011
80	- .7625102666-002	.1526116271+001	- .1076337320+001	.2460692709+013	- .1442024426+011
81	- .75423501911-002	.156615921+001	- .10753738465+001	.26426311035+013	- .1574646362+011
82	- .74914323063-002	.143671440+001	- .107443395+001	.2842632513+013	- .1671342321+011
83	- .7297134433-002	.146276465+001	- .1073631236+001	.3050796325+013	- .1773779433+011
84	- .71909140296-002	.1449291431+001	- .1072641456+001	.3222445253+013	- .18106290361011
85	- .7093520799-002	.143122667+001	- .107172364+001	.3506532662+013	- .1923665531011
86	- .6995325301-002	.144326125+001	- .1070924104+001	.37552373524+013	- .2109774524+011
87	- .6900341191-002	.1439655663+001	- .1070935654+001	.4018244610+013	- .2312427795+011
88	- .6893323505-002	.14329323505+001	- .10692863394+001	.42963435101+013	- .2560727213+011
89	- .6714024360-002	.13343237930+001	- .1063956493+001	.46903037063+013	- .349465603631011
90	- .6630451624-002	.1347521195+001	- .1067222494+001	.49609564600+013	- .26385054952+011
91	- .6544916610-002	.1331193172+001	- .1066966746+001	.5223061319+013	- .328130419756+011
92	- .6464536760-002	.1331669324+001	- .10562227676+001	.56244636945+013	- .39346563191011
93	- .6329229525-002	.1330360924+001	- .1055037430+001	.59396434625+013	- .46236569043+011
94	- .62009223034-002	.1235225262+001	- .10542292414+001	.63424210023+013	- .52369092693+011
95	- .6203541146-002	.1223014725+001	- .1064105199+001	.67229263469+013	- .54333552426+011
96	- .6143916103-002	.1239909392+001	- .1063422612+001	.7155205155+013	- .56140161324+011
97	- .6024032355-002	.1234542111+001	- .1063264303+001	.7604435415+013	- .59923137235+011
98	- .6022386331-002	.1235213962+001	- .1062114523+001	.80926324332+013	- .639036035992+011
99	- .591195313-002	.121909036+001	- .1061433164+001	.8522462229+013	- .62622914332+011
100	- .5856332776-002	.120631431+001	- .1050354691+001	.9293699699+013	- .64446112330+011
101	- .579650723241-002	.119206305+001	- .1050243745+001	.96410720227+013	- .6636211944+011
102	- .57209326356-002	.1160154944+001	- .1021564176+013	- .4364775302+011	

Table 22. (Concluded).

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$\eta_{r,1}(100)$	$J_r(1)$
0	(.76519768564000)				
1	-97041280054002				
2	-305019430364004				
3	.3941-1972243008				
4	-3207e3201440022				
5	.755610440226002				
6	-1673122044002				
7	.27595170544001				
8	-35233602034000				
9	.3220653007-001				
10	.6304312430-002				
20	.2568484660-015				
29	.2422501969-015				
39	.414231123161-016				
50	.6042565725-016				
60	.42321936092-016				
70	.2164245265-109				
80	.2135501959-130				
90	.2442405549-152				
91	.1636300531-154				
92	.8331232050-152				
93	.65702024344-159				
94	.3726277366-161				
95	.1511261394-160				
96	.6512910014-163				
97	.2611236225-163				
98	.2502923120-129				
99	.16417438314-122				
100	.1229233117-125				
101	.3240764406-127				
102	.1944406404-129				
103	.1000000103-121				
104	.5023330104-120				
105	.2562235117-125				
106					
107					

Table 23. Computation of $J_r(1)$ from the JYQ equation with $\mu=0$, $x_1=1$, $x_2=0.5$, $i=0$, $m=100$, $j=1$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$\eta_{r,1}(100)$	$P_r(0.5)$
0	(.100000000+001)				
1	(.500000000+000)				
2	(-.125000000+000)				
3	(-.6221934000+002)				
4	(-.1285420626+004)				
5	(.139631029+004)				
6	(.332612329+005)				
7	(.3622006222+005)				
8	(.55266706222+005)				
9	(.5664229204+006)				
10	(-.46542294979+006)				
20	(-1513635454+003)				
30	(.2249062204+009)				
40	(-.2759425964+010)				
50	(-.3322214033+010)				
60	(.3625715210+011)				
70	(-.6240340282+011)				
80	(-.4736312292+011)				
90	(.94033142369+012)				
91	(.2330509365+012)				
92	(-.1034673047+012)				
93	(-.4180469772+012)				
94	(-.3269253023+012)				
95	(.1237592459+012)				
96	(.2937493314+012)				
97	(.3993075042+012)				
98	(-.1424970652+012)				
99	(-.5917231210+012)				
100	(-.4640366238+012)				
101	(.1741476414+012)				
102	(.6085391516+012)				
103	(.64324306674+012)				
104	(.4646223100+012)				
105	(-.32066613141+012)				

Table 24. Computation of $P_r(0.5)$ from the JYQ equation with $\mu=0$, $x_1=1$, $x_2=0.5$, $i=0$, $m=100$, $j=3$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$n_{r,1}(100)$	$Q_r(0,5)$
0	(-.5490061429+000)				
1	(-.7258239278400)				
2	(-.8186632600400)				
3	-.50424764554002				
4	.17161009234004				
5	.1694762754005				
6	.313400626924005				
7	-.11975026234006				
8	-.43119312114006				
9	-.2610002574006				
10	.1246302354007				
20	-.11126664793+009				
30	.62650025386+009				
40	.62291045464010				
50	-.2659749110+011				
60	.1873314620+011				
70	.9317992520+011				
80	-.2369157223+012				
90	.13900283394012				
91	-.4241920103+012				
92	-.62011423164012				
93	-.1792562305+012				
94	.5001373540+012				
95	.7413102303+012				
96	.2130232442+012				
97	-.6050043248+012				
98	-.8310013469+012				
99	-.2541031695+012				
100	.7169419123+012				
101	.1041059452+013	.4707947541-002	.9427932240-001	.6442970616-003	.1294320432+000
102	.2690484912+012	.2195844063-004	.9492421946-001	.8650374303-006	.8461931293-001
103	-.840017042+012	.1012722032-006	.9492503520-001	-.11272575332-007	-.9364043140-001
104	-.123500295+013	.4646273100-009	.9492507402-001	-.7437943037-010	-.1224465835+000
105	-.360495553+012	.2105200759-011	.9492502395-001	-.9756926306-013	.0000000000000000
106					.

Table 25. Computation of $Q_r(0,5)$ from the JYQ equation with $\mu=0$, $x_1=1$, $x_2=0.5$, $i=0$, $m=100$, $j=3$.

Table 26. Computation of $Y_f(1)$ from the JYPQ equation with $\mu=0$, $x_1=1$, $x_2=0.5$, $i=0$, $m=100$, $j=4$.

Example 4.3.2 In this example we compute the solutions $P_r^1(0.5)$ and $Q_r^1(0.5)$ of the homogeneous equation defined in Figure 7, with $\mu = 1$, $x_1 = 1$, $x_2 = 0.5$, for $r = 0, 1, 2, \dots, 100$. The algorithm will be stable provided that we take three initial values for each of these functions; compare (4.3.3). Tables 27 and 28 give the results of these computations for selected values of r ; see §4.0 for a full description of these tables. Initial values were obtained by hand calculation of the explicit elementary forms given in Tables 8.2 and 8.4 of [16]. The computed function values agreed satisfactorily for $r \leq 10$ with the limited data available in [16] and [17]. In addition, satisfactory agreement was obtained with double-precision values produced by an unpublished program of Dr. J. M. Smith; compare Example 4.3.1. Consequently, the stability of the method is confirmed for this example.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$n_{r,1}(100)$	$p_r^1(0.5)$
0	(- .0000000000)				
1	(- .3660254033+000)				
2	(- .129903106+001)				- .32242565270+000
3	(- .163029232+000)				- .1363164693+001
4	(- .4623799694+000)				- .19236256639+000
5	(.51697566124+005)				- .49723692356+000
6	(- .72313316345+005)				- .12109326600+000
7	(- .63651033944+000)				- .2401644462+000
8	(- .2111335446+000)				- .6267643065+000
9	(- .14691369603+000)				- .2009663929+001
10	(- .7692339043+000)				- .3739991924+001
20	(- .138051340+024)				- .11932602120+001
30	(- .593209520+010)				- .38290662288+001
40	(- .1071445334+012)				- .58328147463+001
50	(- .6465401124+012)				- .120392162+001
60	(- .5826731113+012)				- .510616322+001
70	(- .43629335992+013)				- .7425214923+001
80	(- .14409966022+014)				- .2092210162+001
90	(- .3444063043+013)				- .5812021343+001
91	(- .2452069243+010)				- .100+09624343+001
92	(- .3605224276+012)				- .2122230945+001
93	(- .10469227631+014)				- .59901130205+001
94	(- .30349322720+015)				- .100+3661655+001
95	(- .2469563955+016)				- .21616626232+001
96	(- .1239235922+016)				- .100+201213162+001
97	(- .410+293363955+017)				- .219556320266+001
98	(- .100+230361662+017)				- .100+342423042+001
99	(- .64650949310+017)				- .2102222256+011
100	(- .46650001601+017)				- .100+391320000+001
101	(- .65600003021+014)				- .4702495729+002
102	(- .19181119236+014)				- .2195426657+002
103	(- .55600262626+014)				- .10144435054+006
104	(- .8020232656+014)				- .4644445433-009
105	(- .23102324236+014)				- .1102222256+011

Table 27. Computation of $P_r^1(0.5)$ from the JYQ equation with $\mu=1$, $x_1=1$, $x_2=0.5$, $i=0$, $m=100$, $j=3$.

r	$z(r)$	$w_{100}(r)$	$y_r(100)$	$\eta_{r,1}(100)$	$Q_r^1(0.5)$
0	(-.1154700532+001)				
1	(-.1054363845+001)				
2	(-.12061600493+000)				
3	.70589139529+003				
4	.36706721332+006				
5	-.1642316003+005				
6	-.3541824382+006				
7	-.9109288024+006				
8	.26392346295+005				
9	.7331960473+000				
10	.122352300+003				
20	.5550235650+000				
30	-.3321422965+011				
40	-.1707363632+012				
50	.2620542624+012				
60	-.3394426300+013				
70	.6397023322+013				
80	.5997313625+013				
90	-.3904803107+014				
91	-.332747904+014				
92	.1504283999+014				
93	-.15042839164+014				
94	-.47323729561+014				
95	-.1332221546+014				
96	-.7402237343+014				
97	-.5053026938+014				
98	.2379493805+014				
99	.9164912449+014				
100	.2125657602+014				
101	-.2772207200+014	.4707495729-002	.9544135226+001	-.1737295213-001	-.35397430314+001
102	-.1112390222+015	.2195426652-004	.9526762224+001	-.3250313121-003	-.13178424224+002
103	-.3738386345+014	.10144305604-006	.9526437243+001	-.1475296369-005	-.96624153234+001
104	.8337942381+014	.4644545406-009	.9526436067+001	.2077307341-003	.3624579564+001
105	.1842807424+015	.2107227236-011	.9526436039+001	.3773093262-010	.3006656000
106			.9526436069+001		

Table 28. Computation of $Q_r^1(0.5)$ from the JYQ equation with $\mu=1$, $x_1=1$, $x_2=0.5$, $i=0$, $m=100$, $j=3$.

4.4 Examples Involving the Inhomogeneous JYPQ Equation

Using Stirling's formula, we find

$$(4.4.1) \quad \frac{\Gamma(r+\mu+1)}{\Gamma(r+3/2)} \sim r^{\mu-1/2}, \quad r \rightarrow \infty$$

for fixed values of $\mu = 0, 1, 2, \dots$. Comparison of (4.2.1) with (4.3.1), (4.3.2) and (4.4.1) shows that

$$(4.4.2) \quad E_r(x_1) < \begin{cases} P_r^\mu(x_2) \\ Q_r^\mu(x_2) \end{cases}, \quad r \rightarrow \infty$$

for fixed μ , x_1 and x_2 such that $x_1 > 0$ and $x_2 < 1$. Accordingly, the Anger-Weber function is a solution of type 1 and asymptotically the correct number of initial values to be used in every case is $j = 1$; compared (4.3.3). However, the separation of $E_r(x_1)$ from the solutions of type 2 is weak when $\mu = 0$ (being of relative order $r^{-1/2}$ at the odd terms of the sequence and $r^{-3/2}$ at the even terms of the sequence). The separation is also weak when $\mu = 1$ (being of relative order $r^{-3/2}$ at the odd terms of the sequence and $r^{-5/2}$ at the even terms of the sequence). Accordingly, we treat $E_r(1)$ as a solution of type 2 in both cases, even though when $\mu = 1$ greater numerical instability is incurred than in any of the other examples. This is especially true at the even terms of the sequence, which are separated from the Legendre functions by the relative order of $r^{-5/2}$.

From (4.2.4) and (4.3.3) it is seen that the type of the Struve function $H_r(x_1)$ with respect to the JYPQ operator is 0 for all admissible values of x_1 and x_2 . But again because of the weakness of the separation, we regard $H_r(x_1)$ as belonging to type 1.

Example 4.4.1 For the inhomogeneous fourth-order difference equations defined in Figure 7, we compute $E_r(1)$ with $\mu = 1$, $x_1 = 1$ and $x_2 = 0.5$, for $r = 0, 1, 2, \dots, 100$; and $H_r(0.1)$ with $\mu = 1$, $x_1 = 0.1$ and $x_2 = 0.5$, for $r = 0, 1, 2, \dots, 50$. (We restrict the sequence of $H_r(0.1)$ to $r \leq 50$ for the same reasons that were described in Example 4.2.1.) For reasons given above, we use 3 initial values (rather than 1) for $E_r(1)$ and 1 initial value (rather than 0) for $H_r(0.1)$. Initial values for $E_r(1)$ and $H_r(0.1)$ were computed from series expansions given in [21] and checked against values found in [21] and [23].

Tables 29 and 30 give extracts from the results of these computations for selected values of r ; see §4.0 for a full description of these tables. Comparison of the function values produced by the algorithm with values given in [21] and [23] show good agreement over the available ranges, which, however, do not extend beyond $r = 13$. Comparison of the values of $E_r(1)$ in Table 2.9 with values computed to 10 significant figures from the series expansion shows a steady deterioration of precision at the even values of r , culminating in a relative error of $0.5(10)^{-5}$ at $r = 100$. At the odd values of r the precision also deteriorates steadily, but at a slower rate, culminating in a relative error of $0.8(10)^{-7}$ at $r = 99$. These rates of loss of precision are consistent with the separation ratios of $r^{-5/2}$ and $r^{-3/2}$ which govern the growth of the error in this example.

That is, an individual relative rounding error of, say, 10^{-10} in one of the stored initial values of $E_r(1)$ will be amplified by a factor of approximately $100^{5/2} = 10^5$ at $E_{100}(1)$ and by $100^{3/2} = 10^3$ at $E_{99}(1)$. (As noted in Eq. (4.2.1), the function is in behavior different at even and odd values of r .) These amplifications after 100 steps are greater than the factor of 100 that was predicted and observed in the computation of

$A_r(1)$ in Example 4.1.3 and Table 13, where $A_r(1)$ was computed as if it were a function of the next higher type than it actually is.

Finally, comparison of the values of $H_r(0.1)$ in Table 30 with values computed to 10 significant figures from the series expansion shows agreement to 9 or more significant figures at all values of r . The loss of precision in the Struve function computation is slight, due to the separation ratio of only $r^{-1/2}$, but it offsets impossibly slow convergence which would be evidenced if the theoretically correct number of initial values were to be taken.

r	$z(r)$	$v_{100}(r)$	$y_r(100)$	$\eta_{r,1}(100)$	$E_r(1)$
0	(- .5623566270+000) .4231e23162+000)				
1	(- .1717319545+000) .7890924173+002)				
2	.12953230622+003 .9650280362+004)				
3	.139 .321629535+000)				
4	.5179329129+000)				
5	.149316162+000)				
6	.147372326+000)				
7	.97916303226+000)				
8	.157939525+000)				
9	.103263938+000)				
10	.172632624+000)				
20	.3219796520+011)				
30	.1529437920+009)				
40	.697366904+011)				
50	.2062453962+009)				
60	.410325650+011)				
70	.2320947203+009)				
80	.4556129153+011)				
90	.2421221023+009)				
91					
92					
93					
94					
95					
96					
97					
98					
99					
100					
101	.50415245994+011 .2628192263+009)	.4707495729+002	.3215467135+004	.3151220350+004	.630323401+002
102	.5562324637+011 .10252213+009)	.2195426637+004 .1014433501+006)	.6367287435+004 .6367363975+004)	.7649079534+009 .7500491654+009)	.4120234460+004 .6131352e32+004)
103	.6136312994+011 .3066309790+009)	.46446454406+009 .2107227256+011)	.6367433979+004 .6367433983+004)	.1750336505+013 .1717202460+013)	.4336623537+004 .3357270079+013)
104		.9474675270+014)	.6367433983+004 .6367433983+004)	.3357270079+013 .3357270079+013)	.666623537+004 .666623537+004)
105					
106					

Table 29. Computation of $E_r(1)$ from the JYPO equation with $\mu=1$, $x_1=1$, $x_2=0.5$, $i=0$, $m=100$, $j=3$.

r	$z(r)$	$w_{50}(r)$	$y_r(50)$	$\eta_{r,1}(50)$	$H_r(0.1)$
0	(.6259126999-001)				.2120651691-002
1	.1764475402+002				.4242111249-004
2	.3725705347+001				.6060802396-006
3	.2406152537+000				.6734676051-008
4	.1130603957-001				.6122713212-010
5	.29519560360-003				.4709944262-012
6	.5555247554-005				.3140044933-014
7	.005963246-607				.1847123373-016
8	.9421230136-009				.9721864425-019
9	.9181233306-011				.4629523149-021
10	.7641249722-013				.2012859439-023
11	.6535693734-015				.6051517475-026
12	.3544096079-017				.2982063399-028
13	.2030976239-019				.10233307256-030
14	.1051029043-021				.3317141764-033
15	.4960510130-024				.1005200276-035
16	.2149332779-026				.2872015573-038
17	.8609246729-029				.7762240113-041
18	.3203064964-031				.1990326276-043
19	.1112427579-033				.48544726443-046
20	.3621645659-036				
25	.5651366921-049				.2136010241-059
30	.2686669010-062				.3572007851-073
35	.4264654216-076				.2657537517-087
40	.3911351153-090				.9351329257-102
41	.5440692753-093				.1186903261-104
42	.7364236054-096				.1396363919-107
43	.9707556720-099				.1605017331-110
44	.1247075179-101				.1203391300-113
45	.1562559694-104				.1981750779-116
46	.1909659297-107				.2130916437-119
47	.2379042662-110				.2243071557-122
48	.2655649149-113				.2312446509-125
49	.3027453931-116				.2335306112-128
50	.3373306313-119				.2312630801-131
51	.36772334303-122		.8829267290+000		.2223791084-131
52	.3923673771-125		.9595439215-003		.2378697598-140
53	.4099657433-128		-.6907674373+000		-.1939641672-140
54	.4196379447-131		-.6142402122+000		-.1765443724-143
55					.0000000000
56					.0000000000

Table 30. Computation of $H_r(0.1)$ from the JYPO equation with $\mu=1$, $x_1=0.1$, $x_2=0.5$, $i=0$, $m=50$, $j=1$.

CHAPTER 5. SUMMARY AND CONCLUSIONS

This thesis is concerned with the numerical solution of a broad class of linear difference equations of arbitrary order, and of either homogeneous or inhomogeneous form. An algorithm is presented for the stable computation of any solution of such an equation. The results apply directly to scalar linear difference equations but could be adapted to similar problems posed in vector form.

Historically, the algorithm is an outgrowth of algorithms of Miller and Olver. The underlying idea of Miller's algorithm is the following: If a solution $y(r)$ has growth behavior, as $r \rightarrow \infty$, that is less than or equal to the growth behavior of all independent solutions, then in the reverse direction its growth behavior is greater than or equal to the growth behavior of all independent solutions. Consequently, because of the superposition principle, errors introduced in computing $y(r)$ by backward recurrence cannot grow faster than $y(r)$ itself, and in this sense the backward recurrence process is stable. However, this is true only in a region in which the solutions actually maintain the growth behavior indicated by their asymptotic forms as $r \rightarrow \infty$. For this reason we call this the asymptotic region; evidently it depends on the asymptotic structure of the given difference equation for large r .

Miller's algorithm starts the backward recurrence process at a value of r that is larger than any r for which a valid approximation of $y(r)$ is wanted. Arbitrary starting values are assumed. The computed

solution is essentially a scalar multiple of $y(r)$ for early values of r when (i) the difference equation is homogeneous; (ii) $y(r)$ is subdominant compared to every independent solution of the equation. Under these conditions the algorithm is stable and $y(r)$ is obtained from the computed solution for early values of r by a normalization procedure, such as matching the value of the computed solution at $r = 0$ with $y(0)$ in order to determine the correct scale factor. Modified normalization procedures are required when the asymptotic behaviors of more than one fundamental solution are similar.

Miller's algorithm involves no forward recurrence, and is stable only for solutions whose growth behavior is less than or equal to the growth behavior of all independent solutions. Olver's algorithm was applied originally to second-order inhomogeneous difference equations, in cases where a particular solution $y(r)$ has growth behavior as $r \rightarrow \infty$ that lies between that of a pair of linearly independent solutions of the homogeneous equation. To construct a stable algorithm, Olver posed a boundary value problem having one condition at each end. The initial condition is the value of $y(r)$ at a suitable point $r = i$. The terminal condition is zero, it being anticipated that isolated values of $y(r)$ for large r will not be easily obtained. The resulting boundary value problem is a tri-diagonal linear system of algebraic equations. Olver specified a method for solving this linear system that involves a forward recurrence stage and a backward recurrence stage. The method is equivalent to Gaussian elimination without pivoting, scaled in a

particular way. Analysis of the method shows that (i) the forward and backward recurrence stages are stable, at least in ranges beginning at a sufficiently large value of r ; (ii) the second-order linear difference operator that corresponds to the difference equation is, in effect, factored into the product of two first-order operators. Furthermore Olver derived an infinite series expansion of the truncation error, valid at any point r , that is incurred by choosing a particular terminal point for the boundary value problem. This enabled the optimal terminal point to be calculated automatically, in contrast to Miller's algorithm in which the terminal point is guessed and subsequently tested.

Extensions of Olver's algorithm to difference equations of higher order were proposed by Zahar, Oliver and Cash. Criteria for selecting the correct number of initial conditions (and therefore the correct number of terminal conditions, since the total number of conditions must equal the order of the difference equation) were given independently by Zahar and Oliver. Zahar proved convergence of the algorithm under assumed conditions on the solutions of the adjoint equation. Oliver investigated stability by analyzing the factorization of the finite boundary value problem. Cash provided an extension of the series expansion of the truncation error. The work of Zahar, Oliver and Cash, taken together, contains the essential elements of a practicable generalization of Olver's algorithm.

In this thesis Olver's algorithm has been extended and analyzed afresh. The view was taken that a linear difference operator is an

infinite upper-triangular band matrix. For a specified initial point i and number of initial conditions j , an (i,j) -factorization of a linear difference operator D is said to exist if Gaussian elimination without pivoting about the j -th super-diagonal of D , starting at the point i on this diagonal, can be carried out indefinitely. This process produces two lower-order difference operators (infinite band matrices). Their product is, of course, the original operator, and the sum of their orders is the order of the original operator. Furthermore, the solution of any finite boundary-value problem having initial point i , number of initial points j and terminal point n may be found by forward recursion with one of these lower-order operators followed backward recursion with the other (Theorem 1.2.2). Procedurally, a finite boundary value problem is solved by the algebraic method of forward elimination without pivoting followed by back substitution; this is accompanied by an analysis of the stability of this process in terms of linear difference operators.

Determination of the correct number of initial conditions for stability requires a knowledge of the relative rates of asymptotic growth of solutions of the homogeneous equation as $r \rightarrow \infty$. Since the number of linearly independent homogeneous solutions is equal to the order ℓ of the linear difference operator, there cannot be more than ℓ distinct growth rates as $r \rightarrow \infty$. Cases in which there are exactly ℓ distinct growth rates are the easiest to treat. In these cases a basis exists that is linearly ordered by growth rate, and we call the operator totally separable. However, there exist operators of practical

interest, including constant-coefficient operators for which one or more of the distinct characteristic roots are equal in magnitude, that are not totally separable. Accordingly, a more general class of operators, called separable operators, is introduced (Definition 2.2.1). The solutions of a homogeneous equation that has a separable, but not totally separable, operator exhibit less than ℓ distinct rates of growth as $r \rightarrow \infty$. Separable operators are characterized by the existence of certain bases, called optimally ranked. An optimally ranked basis includes $\sigma \leq \ell$ disjoint subsets such that (i) neither of two distinct solutions in the same subset dominates the other as $r \rightarrow \infty$; (ii) any two solutions taken from distinct subsets are such that one dominates the other; (iii) no nonzero linear combination of solutions from one subset is dominated by any solution in the subset. These conditions ensure that separable operators are unambiguously defined in the sense that any two distinct optimally ranked bases have similar structure (Lemma 2.2.1).

The disjoint subsets of an optimally ranked basis are arranged in linear order by increasing rate of growth. All possible solutions of an inhomogeneous difference equation with arbitrary right side are classifiable by comparison with the solutions in an optimally ranked basis of the corresponding homogeneous form. Thus a particular solution $y(r)$ is said to be of type s if (i) $y(r)$ is not dominated as $r \rightarrow \infty$ by any solution in the first s subsets of an optimally ranked basis; (ii) $y(r)$ is dominated by every solution in the remaining $\sigma-s$ subsets. The extreme cases $s=\sigma$ and $s=0$ correspond to particular

solutions for which forward recurrence and backward recurrence of the given difference equation are stable, respectively. For the intermediate cases $0 < s < \sigma$, let j be the number of solutions in the first s subsets. If in addition to being separable the difference operator is (i,j) -factorizable for some initial point i , analysis of the boundary value problem shows that the equivalent pair of forward and backward recurrence problems (Theorem 1.2.2) are stable in the sense that rounding errors are not propagated more rapidly than the solutions, at least in the asymptotic region belonging to the given problem. Accordingly, the solution $y(r)$ of the boundary value problem is obtained in a stable manner by forward elimination without pivoting followed by back substitution (§3.3 and Theorem 3.3.1). Furthermore, under appropriate conditions it follows that (i) $y(r)$ is uniquely determined by j initial values (Theorem 3.1.1); (ii) the algorithm converges pointwise as the terminal point proceeds to infinity (Theorem 3.2.1).

The numerical examples included in this thesis were chosen so that known asymptotic results could be used in conjunction with the new classification theory for separable operators. This enabled the correct number of initial conditions for any chosen solution to be ascertained immediately. The examples also show that it is sometimes desirable to associate a solution with solutions of the next higher theoretical type. This artificial raising of the type is manifested in the algorithm by the specification of the number of initial conditions appropriate for the higher type. This artifice preserves unique determination

of the solution and also convergence of the algorithm, but introduces instability. It is employed only when the asymptotic separation between two types is weak, consequently the loss of precision resulting from the instability is mild. The compensating advantage is that the rate of convergence is increased very substantially, enabling a much smaller terminal point to be used.

The trade-off between numerical instability and slow convergence can be regarded heuristically in the following way. First, suppose $u(r) \sim e^{-r}$ and $v(r) \sim e^r$ are two fundamental solutions of a second-order equation. Then one initial value of $u(r)$ determines the function uniquely and the algorithm is both convergent and stable. Furthermore, the rate of convergence is very rapid. This can be seen from the equation

$$u_n(r) = u(r) - \frac{e^{-n}}{e^n} \cdot v(r)\{1+o(1)\}, \quad n \rightarrow \infty,$$

where $u_n(r)$ is the solution of the approximating boundary value problem having n as its terminal point (compare Eq. 3.2.6). The rapid convergence of $u_n(r)$ to $u(r)$ as $n \rightarrow \infty$ for fixed r is readily apparent. Thus the algorithm performs very well with regard to both numerical stability and rapidity of convergence when the separation of solutions is strong as in this example of exponential separation.

Next, suppose that $u(r) \sim r^{-1}$ and $v(r) \sim r^{-1/2}$. Here again, one initial value of $u(r)$ determines the function uniquely and the algorithm is both convergent and stable. But in this case

$$u_n(r) = u(r) - \frac{n^{-1}}{n^{-1/2}} \cdot v(r)\{1+o(1)\}, \quad n \rightarrow \infty$$

and it is apparent that the rate of convergence will be governed by the very slowly decreasing function $n^{-1/2}$. Suppose, however, that we give an additional initial value of $u(r)$. With two initial values our algorithm is equivalent to forward recurrence of the given difference equation and all questions of convergence disappear. This forward recurrence is, of course, unstable for the desired solution $u(r)$ but the instability is very weak; any small component of the more dominant solution $v(r)$ can grow no faster than $r^{1/2}$ relative to $u(r)$. Although this rate of growth (or any other positive rate of growth) is ultimately sufficient to obliterate the desired solution when computed in finite precision arithmetic, subsequences of the solution of considerable length can be computed before this happens. Thus in many practical situations where the separation of solutions is weak, a minor modification[†] of the algorithm performs acceptably well with regard to both stability and convergence.

Finally, suppose that the relative separation of $u(r)$ and $v(r)$ is like $r^{-3/2}$ or $r^{-5/2}$ (compare Example 4.4.1 which presents the computation of a function exhibiting both forms of behavior, $r^{-3/2}$ on the subsequence of odd terms and $r^{-5/2}$ on the subsequence of even terms). These separation ratios are too weak to allow rapid convergence of the unmodified algorithm, and at the same time too strong for the numerical instability in the modified algorithm to be truly insignificant. Computation of cases such as this are the most difficult for the algorithm to perform well, and performance can be improved only by recourse to extended-precision computation.

[†]In general the modified algorithm will require a back substitution as well as a forward elimination stage. Nevertheless, similar considerations govern the trade-off between numerical instability and slow convergence.

Perhaps it should be stressed that a complete knowledge of the asymptotic behavior of all of the solutions of the homogeneous equation is essential to determine the correct number of initial conditions. This information may be developed for a broad class of linear difference equations using known analytical results (§2.2). An alternative would be to attempt to construct the essential classification parameters - the number of distinct types of solution, and the number of linearly independent solutions of each type - by numerical experimentation. Under this approach, the algorithm presented in this thesis would be applied with chosen initial conditions appropriate to different possible realizations of the classification theory, that is, different possible forms of the optimally ranked basis. Presumably, incorrect choices would be revealed by discernible numerical instability. The effectiveness of such an approach has not been assessed, however.

In conclusion, the results presented in this thesis are distinguished from the work of earlier authors in the following ways. First, the interplay between linear difference operators and infinite band matrices achieves a fuller blending of analytic and algebraic ideas. This leads to a clearer understanding of stability, including the tradeoff choices that exist between slow convergence and mild instability when the asymptotic separation of adjacent solutions is weak. Secondly, the classification theory for separable operators is believed to be new. It enables the proper number of initial conditions that are prescribed by the convergence theorem to be determined directly from the asymptotic behavior of fundamental solutions of the given difference equation. In contrast, Zahar employs fundamental solutions of the adjoint equation and Oliver applies a

point-by-point growth criterion between pairs of solutions. Finally, the numerical examples presented here have been chosen in such a way that the solutions exhibit a much richer variety of growth rates than in any previous investigation.

APPENDIX

Proof of Lemma 2.1.1

The proof consists of (i) identifying the sequence defined by equation (2.1.4) as an almost periodic sequence, and (ii) proving that every nonzero almost periodic sequence has an infinite subsequence which is bounded away from zero. We begin by presenting the essential elements of the theory of almost periodic functions.

First we give the characterization of almost periodic functions that is taken as the definition by Corduneanu [7]. A function of the form

$$T(x) = \sum_{k=1}^n c_k e^{i\lambda_k x}, \quad -\infty < x < \infty,$$

where the c_k are complex numbers, the λ_k are real numbers and $i = \sqrt{-1}$, is called a complex trigonometric polynomial. A function $f(x)$ taking complex values and having all real numbers as domain is an almost periodic function provided that to each $\varepsilon > 0$ there corresponds some trigonometric polynomial $T_\varepsilon(x)$ such that

$$|f(x) - T_\varepsilon(x)| < \varepsilon$$

uniformly on $-\infty < x < \infty$. Clearly, every complex trigonometric polynomial is itself an almost periodic function.

An alternative characterization is the definition of H. Bohr [4]: A continuous complex valued function $f(x)$ defined on the real line is almost periodic if to each $\varepsilon > 0$ there corresponds a positive real number $\delta = \delta(\varepsilon)$ such that every open interval of the real line of length δ contains at least one point ξ for which

$$|f(x+\xi) - f(x)| < \varepsilon$$

uniformly for every x . Any number ξ which satisfies this condition is called a translation number of f corresponding to ε , or more briefly, an ε -translation number of f .

An ε -translation number of f is "almost" a period of f . Every periodic function is almost periodic as well. If p is the fundamental period of a periodic function f , then rp for $r = 0, \pm 1, \pm 2, \dots$ are translation numbers of f for each ε . Bohr's definition generalizes the fact that the periods of a periodic function form an arithmetic progression.

Bohr's definition and Corduneanu's definition define exactly the same class of functions. This is the "fundamental theorem" of the theory of almost periodic functions. See either [4] or [7] for a full account.

Turning now to sequences, we have the following definition presented in [7, §1.6]: A function $f(r)$, taking complex values and having the set of integers as domain, is an almost period sequence provided that to each $\varepsilon > 0$ there corresponds some positive integer $N = N(\varepsilon)$ such that, within any set of N consecutive integers, there exists an integer p for which

$$|f(r+p) - f(r)| < \varepsilon.$$

uniformly over all integers r . Corduneanu proves "A necessary and sufficient condition for a sequence $\{a_r\}$ to be almost periodic is the existence of an almost periodic function $f(x)$, $-\infty < x < \infty$, such that $a_r = f(r)$, $r = 0, \pm 1, \pm 2, \dots$ "; see [7, Theorem 1.27].

We are now ready to prove Lemma 2.1.1. If r were allowed to assume all real values, instead of being restricted to $r = 0, 1, 2, \dots$, then equation (2.1.4) defines a trigonometric polynomial. By Corduneanu's definition, every trigonometric polynomial is an almost periodic function.

By the theorem quoted above, the sequence $x(r)$, $r = 0, \pm 1, \pm 2, \dots$

defined by equation (2.1.4) is an almost periodic sequence. The only thing remaining is to prove the following assertion:

If an almost periodic sequence $\{f(r)\}$ is not identically zero, then there exists an infinite subsequence of $\{f(r)\}$ which is bounded away from zero.

Proof Assume $\{f(r)\}$ is not identically zero. Let r_0 be such that $f(r_0) \neq 0$ and put $A = |f(r_0)|$. Put $\varepsilon = A/(2+A)$. Then $\varepsilon \in (0,1)$, and by the definition of an almost periodic sequence there exists a positive integer N_1 , say, such that for some integer $p_1 \in \{1, 2, \dots, N_1\}$ the inequality

$$|f(r_0 + p_1) - f(r_0)| < \varepsilon$$

is satisfied. Put $r_1 = r_0 + p_1$. Next, there exists a positive integer N_2 such that for some integer $p_2 \in \{1, 2, \dots, N_2\}$ the inequality

$$|f(r_1 + p_2) - f(r_1)| < \varepsilon^2$$

is satisfied. Put $r_2 = r_1 + p_2$. Continuing in this manner, we see that for each $k = 3, 4, 5, \dots$ we can find an integer $r_k > r_{k-1}$ such that

$$|f(r_k) - f(r_{k-1})| < \varepsilon^k.$$

For each $k = 1, 2, 3, \dots$ we have

$$\begin{aligned} |f(r_k) - f(r_0)| &\leq \sum_{j=1}^k |f(r_j) - f(r_{j-1})| \\ &< \sum_{j=1}^k \varepsilon^j < \frac{\varepsilon}{1-\varepsilon} < \frac{1}{2} A . \end{aligned}$$

Since $A = |f(r_0)|$, we have

$$\begin{aligned}|f(r_k)| &= |f(r_0) + f(r_k) - f(r_0)| \\ &\geq A - |f(r_k) - f(r_0)| > \frac{1}{2}A,\end{aligned}$$

which finishes the proof of the assertion. 

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